

# The Cauchy Problem for nonlinear Quadratic Interactions of the Schrödinger type in one dimensional space

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## Abstract

We study well-posedness for the Cauchy Problem associated to the coupled Schrödinger equations with quadratic nonlinearities, which appears modeling problems in nonlinear optics. We obtain local well-posedness for data in Sobolev spaces with low regularity. To develop the local theory, we prove new bilinear estimates for the coupling terms of the system in the continuous case. Concerning global results, in the continuous case, we establish global well-posedness in  $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ , for some negative  $s$ . The proof of our global result uses the **I**-method introduced by Colliander, Keel, Staffilani, Takaoka and Tao.

## 1 Introduction

This work is dedicated to the study of the Cauchy Problem for a system that appears modeling some problems in the context of nonlinear optics. More precisely, we will study the following mathematical model:

$$(1.1) \quad \begin{cases} i\partial_t u(x, t) + p\partial_x^2 u(x, t) - \theta u(x, t) + \bar{u}(x, t)v(x, t) = 0, & x \in \mathbb{R}, t \geq 0, \\ i\sigma\partial_t v(x, t) + q\partial_x^2 v(x, t) - \alpha v(x, t) + \frac{a}{2}u^2(x, t) = 0, \\ u(x, 0) = u_0(x), & v(x, 0) = v_0(x), \end{cases}$$

where  $u$  and  $v$  are complex valued functions and  $\alpha$ ,  $\theta$  and  $a := 1/\sigma$  are real numbers representing physical parameters of the system, where  $\sigma > 0$  and  $p, q = \pm 1$ . The model (1.1) is given by the nonlinear coupling of two dispersive equations of Schrödinger type through the quadratic terms

$$(1.2) \quad N_1(u, v) = \bar{u} \cdot v \quad \text{and} \quad N_2(u, v) = \frac{1}{2}u \cdot v.$$

Physically, according to the article [15], the complex functions  $u$  and  $v$  represent amplitude packets of the first and second harmonic of an optical wave, respectively. The values of  $p$  and  $q$  may be 1 or  $-1$ , depending on the signals provided between the scattering/diffraction ratios, and the positive constant  $\sigma$  measures the scaling/diffraction magnitude indices. In recent years, interest in nonlinear properties of optical materials has attracted the attention of physicists and mathematicians. Many researches suggest that by exploring the nonlinear reaction of the matter, the bit-rate capacity of optical fibers can be considerably increased and in consequence an improvement in the speed and economy of data transmission and manipulation. Particularly in non-centrosimetric materials, those having no inversion symmetry at molecular level, the nonlinear effects of lower order give rise to second order susceptibility, which means that the nonlinear response to the electric field is quadratic; see, for instance, the articles [12] and [9].

Another application for system (1.1) system is related to the Raman amplification in a plasma. The study of laser-plasma interactions is an active area of interest. The main goal is to simulate nuclear fusion in a laboratory. In order to simulate numerically these experiments, we need some accurate models. The kinetic ones are the most relevant but very difficult to deal with practical computations. The fluids ones like bifluid Euler–Maxwell system seem more convenient but still inoperative in practice because of the high frequency motion and the small wavelength involved in the problem. This is why we need some intermediate models that are reliable from a numerical viewpoint [3].

In the mathematical context N. Hayashi, T. Ozawa, K. Tanaka in [11] obtained local well-posed for the Cauchy problem (1.1) on the spaces  $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$  for  $n \leq 4$  and  $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$  for  $n \leq 6$ . In [14] was revised the time decay estimates of small solutions to the systems under the mass resonance condition in 2-dimensional space. They authors also showed the existence of wave operators and modified wave operators of the systems under some mass conditions in  $n$ -dimensional space, where  $n \geq 2$ , and showed the existence of scattering operators and finite time blow-up of the solutions for the systems in higher space dimensions.

With regard to qualitative properties of Cauchy Problem solutions (1.1), we know that in the case where  $p = q = 1$  the system was studied by F. Linares and J. Angulo in [1] for initial data  $u_0, v_0$  in the same periodic Sobolev space  $H^s(\mathbb{T})$ . More precisely, they obtained local well-posedness results in  $H^s(\mathbb{T}) \times H^s(\mathbb{T})$  for all  $s \geq 0$  and obtained global well-posedness in the space  $L^2(\mathbb{T}) \times L^2(\mathbb{T})$  using the conservation of the mass by the flow of the system, that is, the following conservation law:

$$(1.3) \quad E(u(t), v(t)) = \int_{-\infty}^{+\infty} (|u|^2 + 2\sigma|v|^2) dx = E(u_0, v_0).$$

**Remark 1.** *The authors also observed in Comment 2.3 of [1] that results can be obtained for data with lower regularity when  $\sigma$  is different from 1, including:*

well-posedness in  $H_{per}^s \times H_{per}^s$  for  $s > -1/2$ . Furthermore, in the same work, stability and instability results were established for certain classes of periodic pulses. Another work devoted to the study of the existence and stability of wave type pulses for this model is due to A. Yew (see [17]).

The technique used in [1] to obtain the results of local well-posedness follow the ideas of [13], developed by C. Kenig, G. Ponce and L. Vega, where the study of initial value problem for a Schrödinger equation with quadratic nonlinearities in both periodic and continuous domain. More precisely, they considered the following initial value problem:

$$(1.4) \quad \begin{cases} iu_t + \partial_x^2 u = N_j(u, \bar{u}), & x \in \mathbb{R} \text{ or } x \in \mathbb{T}, t \geq 0, \\ u(x, 0) = u_0(x), \end{cases}$$

where  $N_1(u, \bar{u}) = u\bar{u}$ ,  $N_2(u, \bar{u}) = u^2$  and  $N_3(u, \bar{u}) = \bar{u}^2$ . The authors considered initial data in the Sobolev space  $H^s$ . In the continuous case, they proved local well-posedness for  $s > -1/4$  in the case  $j = 1$  and for  $s > -3/4$  in the cases  $j = 2, 3$ . In the periodic case, was obtained local well-posedness for  $s \geq 0$  when  $j = 1$  and for  $s > -1/2$  when  $j = 2, 3$ . To prove the local theory, they used the Fourier restriction norm method, known in the literature  $X^{s,b}$ -spaces and introduced by J. Bourgain in [2]. Within this functional space, sharp bilinear estimates were proved which combined with the Banach Fixed Point Theorem applied to the integral operator associated to (1.1) allowed to obtain the desired local solutions. The lack of a conservation law for (1.4) does not allow global results to be obtained in some space as usually is done.

We note that the results given in [13] can be applied to the system (1.1) in the case where  $\sigma = 1$ . In this situation, it is not difficult to obtain the local well-posedness in  $H^s \times H^s$  for  $s > -1/4$ , however a question arises naturally:

*What will be the scene of the local and global well-posedness of the system (1.1) when  $\sigma \neq 1$  and for initial data taken in Sobolev spaces not necessarily with the same regularity?*

In order to answer the previous question, we consider in this work the Cauchy Problem (1.1) with any  $\sigma > 0$  and initial data  $(u_0, v_0)$  belonging to Sobolev spaces of the form  $H^\kappa(\mathbb{R}) \times H^s(\mathbb{R})$ . As far we know, the local well-posedness for the system (1.1) in low regularity it's unknow. However, as pointed out before, in work [11] was obtained local well-posedness in the space  $H^1(\mathbb{R}) \times H^1(\mathbb{R})$  and  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ . To obtain our results will be very useful the ideias developed by A. J. Corcho and C. Matheus in [8], where they treated the Schrödinger-Debye system, modelled by

$$(1.5) \quad \begin{cases} iu_t + \frac{1}{2}\partial_x^2 u = uv, & x \in \mathbb{R}, t \geq 0, \\ \mu v_t + v = \pm |u|^2, & \mu > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \end{cases}$$

which also has quadratic type nonlinearities and the authors developed a local and global theory in Sobolev spaces with different regularities. The method

employed by them is also based on obtaining sharp bilinear estimates for the coupling terms in suitable Bourgain spaces as well as the use of fixed point techniques. Also, in the same work, global results were obtained via a technique known as **I**-method which was first implemented by J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao in [4].

Before of enunciate the principal results, we given the follows

**Definition 1.** *Given  $\sigma > 0$ , we say that the Sobolev indice pair  $(\kappa, s)$  verifies the hypotheses  $H_\sigma$  if it satisfies one of the following conditions:*

- a)  $|\kappa| - 1/2 \leq s < \min\{\kappa + 1/2, 2\kappa + 1/2\}$  for  $0 < \sigma < 2$ ;
- b)  $\kappa = s \geq 0$  for  $\sigma = 2$ ;
- c)  $|\kappa| - 1 \leq s < \min\{\kappa + 1, 2\kappa + 1\}$  for  $\sigma > 2$ .

We put

$$(1.6) \quad \mathcal{W}_\sigma := \left\{ (\kappa, s) \in \mathbb{R}^2; (\kappa, s) \text{ verify the hypothesis } H_\sigma \right\}.$$

In what follows  $\psi$  denote a cut-off function in  $C_0^\infty$  such that  $0 \leq \psi(t) \leq 1$ ,

$$(1.7) \quad \psi(t) = \begin{cases} 1, & \text{if } |t| \leq 1 \\ 0, & \text{if } |t| \geq 2 \end{cases}$$

$$\text{and } \psi_T(t) = \psi\left(\frac{t}{T}\right).$$

The local well-posedness result is as follows.

**Theorem 1.** *Given  $\sigma > 0$ . Then for any  $(u_0, v_0) \in H^\kappa \times H^s$  with the Sobolev indice pair  $(\kappa, s)$  verifies the hypotheses  $H_\sigma$ , there exist a positive time  $T = T(\|u_0\|_{H^\kappa}, \|v_0\|_{H^s})$  and a unique solution  $(u(t), v(t))$  for the initial value problem (1.1), satisfying*

$$(1.8) \quad \psi_T(t)u \in X^{\kappa, \frac{1}{2}+} \quad \text{and} \quad \psi_T(t)v \in X_{1/\sigma}^{\kappa, \frac{1}{2}+},$$

$$(1.9) \quad u \in C([0, T]; H^\kappa(\mathbb{R})) \quad \text{and} \quad v \in C([0, T]; H^s(\mathbb{R})).$$

Moreover, the map  $(u_0, v_0) \mapsto (u(t), v(t))$  is locally Lipschitz from  $H^\kappa(\mathbb{R}) \times H^s$  into  $C([0, T]; H^\kappa(\mathbb{R}) \times H^s)$ .

Concerning global well-posedness we have the following result.

**Theorem 2.** *The Cauchy Problem associated to the system 1.1 is Global well-posed, that is, it has a unique local-in-time solution with initial conditions  $(u_0, v_0) \in H^s \times H^s$ . In the following cases:  $\sigma = 2$  and  $s = 0$ ;  $\sigma > 2$  and  $s \geq -1/2$  and in the case  $0 < \sigma < 2$  and  $s \geq -1/4$ .*

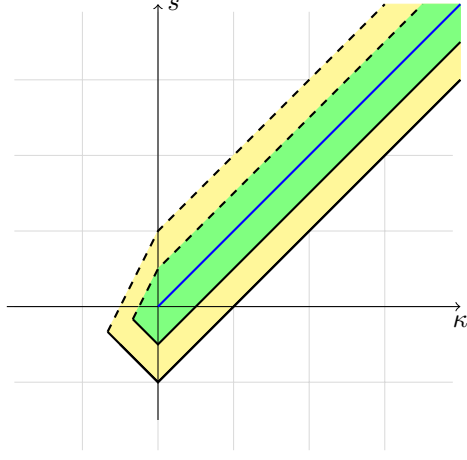


Figure 1: Blue: case  $\sigma = 2$ , Green: case  $0 < \sigma < 2$ , Yellow: case  $\sigma > 2$ .

This work is structured as follows. The Section 2 is devoted to summarize some preliminary results. In Section 3, we will develop a local theory in Bourgain spaces, following closely the techniques used in [13] and [8], where for each positive  $\sigma$  we obtain quite general results in Sobolev spaces with regularities out of the diagonal case  $\kappa = s$ . Specifically, we will prove local well-posedness for data  $(u_0, v_0) \in H^\kappa \times H^s$  with indices  $(\kappa, s) \in \mathcal{W}_\sigma$  where the flat region  $\mathcal{W}_\sigma$ .

Finally, in Section 4 we will use the I-method to extend globally the local solutions obtained for data in  $H^s \times H^s$  with  $s \leq 0$ . Specifically, for regularity  $-\frac{1}{4} \leq s \leq 0$  when  $0 < \sigma < 2$  and  $-\frac{1}{2} \leq s \leq 0$  when  $\sigma > 2$ . At this point, it will be crucial, the use of a refined Strichartz-type estimate in Bourgain's spaces for the Schrödinger equation. For details about this estimate we refer [5].

## 2 Preliminary results

We consider the equation of the form

$$(2.1) \quad i\partial_t \omega - \phi(-i\partial_x) \omega = F(\omega),$$

where  $\phi$  is a measurable real-valued function and  $F$  some nonlinear function.

The Cauchy Problem for (2.1) with initial data  $\omega(0) = \omega_0$  is rewritten as the integral equation

$$(2.2) \quad \omega(t) = W_\phi(t)\omega_0 - i \int_0^t W_\phi(t-t')F(\omega(t'))dt',$$

where  $W_\phi(t) = e^{-it\phi(-i\partial_x)}$  is the group that solves the linear part of (2.1).

Let  $X^{s,b}(\phi)$  be the completion of  $\mathcal{S}(\mathbb{R}^2)$  with respect to the norm

$$\begin{aligned}
\|f\|_{X^{s,b}(\phi)} &:= \|W_\phi(-t)f\|_{H_t^b(\mathbb{R}, H_x^s)} \\
(2.3) \quad &= \left\| \langle \xi \rangle^s \langle \tau \rangle^b \mathcal{F} \left( e^{it\phi(-i\partial_x)} f \right) (\tau, \xi) \right\|_{L_\tau^2 L_\xi^2} \\
&= \left\| \langle \xi \rangle^s \langle \tau + \phi(\xi) \rangle^b \widehat{f}(\tau, \xi) \right\|_{L_\tau^2 L_\xi^2}.
\end{aligned}$$

Before stating the results, we will give some useful notations.

The following lemma has been proved while establishing the local well-posedness of the Zakharov system by Ginibre, Tsutsumi and Velo in [10].

**Lemma 1.** *Let  $-\frac{1}{2} < b' \leq 0 \leq b \leq b' + 1$ ,  $\psi$  a cutoff function and  $T \in [0, 1]$ . Then for  $F \in X^{s,b'}(\phi)$  we have*

$$(2.4) \quad \|\psi_1(t)W_\phi(t)\omega_0\|_{X^{s,b}(\phi)} \leq C \|\omega_0\|_{H^s},$$

$$(2.5) \quad \left\| \psi_T(t) \int_0^t W_\phi(t-t')F(\omega(t'))dt' \right\|_{X^{s,b}(\phi)} \leq CT^{1+b'-b} \|F\|_{X^{s,b'}(\phi)}.$$

*Proof.* See Lemma 2.1 in [10].  $\square$

In our case we shall use the space  $X^{s,b}(\phi)$  for the phase functions  $\phi_1(\xi) = \xi^2$  and  $\phi_a(\xi) = a\xi^2$ . Indeed we can rewrite the system (1.1) in the form

$$(2.6) \quad \begin{cases} i\partial_t u - \phi_1(-i\partial_x)u - \theta u + \bar{u}v = 0, \\ i\partial_t v - \phi_a(-i\partial_x)v - \alpha v + \frac{a}{2}u^2 = 0, \quad a > 0. \end{cases}$$

Then we have

$$X^{\kappa,b}(\phi_1) = X^{\kappa,b}, \quad W_{\phi_1} = U(t) = e^{it\partial_x^2}$$

and

$$X^{s,b}(\phi_a) = X_a^{s,b}, \quad W_{\phi_a} = U_a(t) = e^{iat\partial_x^2},$$

where  $U(t)$  is the linear Schrödinger group.

We finish this section with the following inequality which will be used to estimate the nonlinear terms in Section 3.

**Lemma 2.** *Let  $p, q > 0$ . Then, for  $r = \min\{p, q\}$  with  $p + q > 1 + r$ , then there exists  $C > 0$  such that*

$$(2.7) \quad \int_{\mathbb{R}} \frac{dx}{\langle x - \alpha \rangle^p \langle x - \beta \rangle^q} \leq \frac{C}{\langle \alpha - \beta \rangle^r}.$$

Moreover, for  $q > \frac{1}{2}$  such that,

$$(2.8) \quad \int_{\mathbb{R}} \frac{dx}{\langle \alpha_0 + \alpha_1 x + x^2 \rangle^q} \leq C \quad \text{for all } \alpha_0, \alpha_1 \in \mathbb{R}.$$

*Proof.* See Lemma 2.3 in [13].  $\square$

### 3 Bilinear estimates for the coupling terms

We begin this section by enunciating two following lemmas, which will be used to estimate the bilinear functionals.

**Lemma 3.** *Let  $b > 1/2$  and  $1/4 < d < 1/2$ . To prove the inequality*

$$(3.1) \quad \|\overline{u} \cdot v\|_{X^{\kappa, -d}} \leq c \|u\|_{X^{\kappa, b}} \cdot \|v\|_{X_a^{s, b}},$$

*is it enough to prove that the functionals*

$$\begin{aligned} J_1 &= \frac{1}{\langle \tau + \xi^2 \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|} \chi_{\mathcal{R}_1}}{\langle \tau - (a-1)\xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2; \\ J_2 &= \frac{1}{\langle \tau_2 + a\xi_2^2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|} \chi_{\mathcal{R}_2}}{\langle \tau_2 + 2\xi^2 + \xi_2^2 - 2\xi\xi_2 \rangle^{2d}} d\xi; \\ J_3 &= \frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|} \chi_{\mathcal{R}_3}}{\langle \tau_1 - a\xi_2^2 + \xi^2 \rangle^{2d}} d\xi_2, \end{aligned}$$

*are bounded, where  $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 = \mathbb{R}^4$  with  $\mathcal{R}_j$  measurable.*

*Proof.* We define  $f(\xi, \tau) = \langle \tau - \xi^2 \rangle^b \langle \xi \rangle^\kappa \widehat{u}(\xi, \tau)$  and  $g(\xi, \tau) = \langle \tau + a\xi^2 \rangle^b \langle \xi \rangle^s \widehat{v}(\xi, \tau)$ .

Therefore,  $\|f\|_{L_{\xi, \tau}^2} = \|u\|_{X^{\kappa, b}}$  and  $\|g\|_{L_{\xi, \tau}^2} = \|v\|_{X_a^{s, b}}$ .

So,

$$\begin{aligned} \|\overline{u} \cdot v\|_{X^{\kappa, -d}} &= \left\| \langle \tau + \xi^2 \rangle^{-d} \langle \xi \rangle^\kappa \widehat{\overline{u} \cdot v}(\xi, \tau) \right\|_{L_{\xi, \tau}^2} \\ &= \sup_{\|\varphi\|_{L_{\xi, \tau}^2} \leq 1} \left| \int_{\mathbb{R}^4} \frac{\langle \xi \rangle^\kappa \langle \xi_1 \rangle^{-\kappa} \langle \xi_2 \rangle^{-s}}{\langle \tau + \xi^2 \rangle^d \langle \tau_1 - \xi_1^2 \rangle^b \langle \tau_2 + a\xi_2^2 \rangle^b} f(\xi_1, \tau_1) g(\xi_2, \tau_2) \varphi(\xi, \tau) d\xi_2 d\tau_2 d\xi d\tau \right|. \end{aligned}$$

We use the following notation:

$$(3.2) \quad \begin{cases} \tau = \tau_1 + \tau_2 & \xi = \xi_1 + \xi_2 \\ \omega = \tau + \xi^2, & \omega_1 = \tau_1 - \xi_1^2, \quad \omega_2 = \tau_2 + a\xi_2^2 \end{cases}$$

and we define

$$W(f, g, \varphi) = \int_{\mathbb{R}^4} \frac{\langle \xi \rangle^\kappa \langle \xi_1 \rangle^{-\kappa} \langle \xi_2 \rangle^{-s}}{\langle \omega \rangle^d \langle \omega_1 \rangle^b \langle \omega_2 \rangle^b} f(\xi_1, \tau_1) g(\xi_2, \tau_2) \varphi(\xi, \tau) d\xi_2 d\tau_2 d\xi d\tau.$$

Now is sufficient to prove that

$$|W(f, g, \varphi)| \leq c \|f\|_{L^2} \cdot \|g\|_{L^2} \cdot \|\varphi\|_{L^2}.$$

Consider  $\mathbb{R}^4 \subset \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ , where  $\mathcal{R}_j \subset \mathbb{R}^4$  for  $j \in \{1, 2, 3\}$ . We write now

$$W_i = (f, g, \varphi) = \int_{\mathcal{R}_i} \frac{\langle \xi \rangle^\kappa \langle \xi_1 \rangle^{-\kappa} \langle \xi_2 \rangle^{-s}}{\langle \omega \rangle^d \langle \omega_1 \rangle^b \langle \omega_2 \rangle^b} f(\xi_1, \tau_1) g(\xi_2, \tau_2) \varphi(\xi, \tau) d\xi_2 d\tau_2 d\xi d\tau$$

and observe that  $|W| \leq |W_1| + |W_2| + |W_3|$ .

We estimate separately each case. Using the Cauchy-Schwartz and Hölder inequalities and the Fubini's Theorem we obtain

$$\begin{aligned} |W_1|^2 &= \left| \int_{\mathcal{R}_1} \frac{\langle \xi \rangle^\kappa \langle \xi_1 \rangle^{-\kappa} \langle \xi_2 \rangle^{-s}}{\langle \omega \rangle^d \langle \omega_1 \rangle^b \langle \omega_2 \rangle^b} f(\xi_1, \tau_1) g(\xi_2, \tau_2) \varphi(\xi, \tau) d\xi_2 d\tau_2 d\xi d\tau \right|^2 \\ &\leq \|f\|_{L^2} \|g\|_{L^2} \|\varphi\|_{L^2} \left\| \frac{\langle \xi \rangle^{2\kappa}}{\langle \omega \rangle^{2d}} \left( \int_{\mathbb{R}^2} \frac{\langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2s} \chi_{\mathcal{R}_1} d\xi_2 d\tau_2}{\langle \omega_1 \rangle^{2b} \langle \omega_2 \rangle^{2b}} \right) \right\|_{L_{\xi, \tau}^\infty}. \end{aligned}$$

Similarly we obtain

$$|W_2|^2 \leq \|f\|_{L^2} \|g\|_{L^2} \|\varphi\|_{L^2} \left\| \frac{\langle \xi_2 \rangle^{2s}}{\langle \omega_2 \rangle^{2b}} \left( \int_{\mathbb{R}^2} \frac{\langle \xi_1 \rangle^{-2\kappa} \langle \xi \rangle^{2\kappa}}{\langle \omega_1 \rangle^{2b} \langle \omega \rangle^{2d}} \chi_{\mathcal{R}_2} d\xi d\tau \right) \right\|_{L_{\xi_2, \tau_2}^\infty}$$

and

$$|W_3|^2 \leq \|f\|_{L^2} \|g\|_{L^2} \|\varphi\|_{L^2} \left\| \frac{\langle \xi_1 \rangle^{-2\kappa}}{\langle \omega_1 \rangle^{2b}} \left( \int_{\mathbb{R}^2} \frac{\langle \xi \rangle^{2\kappa} \langle \xi_2 \rangle^{-2s}}{\langle \omega \rangle^{2d} \langle \omega_2 \rangle^{2b}} \chi_{\mathcal{R}_3} d\xi_2 d\tau_2 \right) \right\|_{L_{\xi_1, \tau_1}^\infty}.$$

Using Lemma 2.8 and the fact that  $\langle \xi \rangle^{2\kappa} \langle \xi_1 \rangle^{-2\kappa} \leq \langle \xi_2 \rangle^{2|\kappa|}$ , we have the following inequalities:

$$(3.3) \quad \frac{\langle \xi \rangle^{2\kappa}}{\langle \omega \rangle^{2d}} \int_{\mathbb{R}^2} \frac{\langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2s} \chi_{\mathcal{R}_1}}{\langle \omega_1 \rangle^{2b} \langle \tau_2 + a\xi_2^2 \rangle^{2b}} d\xi_2 d\tau_2 \leq \underbrace{\frac{1}{\langle \omega \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|} \chi_{\mathcal{R}_1}}{\langle \tau - (a-1)\xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2}_{J_1}.$$

$$(3.4) \quad \frac{\langle \xi_2 \rangle^{2s}}{\langle \omega_2 \rangle^{2b}} \int_{\mathbb{R}^2} \frac{\langle \xi_1 \rangle^{-2\kappa} \langle \xi \rangle^{2\kappa} \chi_{\mathcal{R}_2}}{\langle \omega_1 \rangle^{2b} \langle \omega \rangle^{2d}} d\xi d\tau \leq \underbrace{\frac{1}{\langle \tau_2 + a\xi_2^2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|} \chi_{\mathcal{R}_2}}{\langle \tau_2 + 2\xi^2 + \xi_2^2 - 2\xi\xi_2 \rangle^{2d}} d\xi}_{J_2}.$$

$$(3.5) \quad \frac{\langle \xi_1 \rangle^{-2\kappa}}{\langle \omega_1 \rangle^{2b}} \int_{\mathbb{R}^2} \frac{\langle \xi \rangle^{2\kappa} \langle \xi_2 \rangle^{-2s} \chi_{\mathcal{R}_3}}{\langle \omega \rangle^{2d} \langle \omega_2 \rangle^{2b}} d\xi_2 d\tau_2 \leq \underbrace{\frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|} \chi_{\mathcal{R}_3}}{\langle \tau_1 - a\xi_2^2 + \xi^2 \rangle^{2d}} d\xi_2}_{J_3}.$$

□

**Lemma 4.** *Let  $b > 1/2$  and  $1/4 < d < 1/2$ . To prove the inequality*

$$(3.6) \quad \|u \cdot \tilde{u}\|_{X_a^{s, -d}} \leq c \|u\|_{X^{\kappa, b}} \cdot \|\tilde{u}\|_{X^{\kappa, b}}$$

*is it enough to prove that the functionals*



$$\begin{aligned}
J_4 &= \frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{S}_1}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2; \\
J_5 &= \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{S}_2}}{\langle \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2 \rangle^{2d}} d\xi; \\
J_6 &= \frac{1}{\langle \lambda_1 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{S}_3}}{\langle \tau_1 + a\xi^2 + \xi_2^2 \rangle^{2d}} d\xi_2,
\end{aligned}$$

are bounded, where  $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 = \mathbb{R}^4$  with  $\mathcal{S}_j$  measurable.

*Proof.* Denote

$$(3.7) \quad \begin{cases} \tau = \tau_1 + \tau_2, & \xi = \xi_1 + \xi_2 \\ \lambda = \tau + a\xi^2, & \lambda_1 = \tau_1 + \xi_1^2, \quad \lambda_2 = \tau_2 + \xi_2^2. \end{cases}$$

The proof follows by same way used in the previous lemma. □

### 3.1 Bilinear estimates for $\sigma > 2$

Next we prove new bilinear estimates in the case  $\sigma > 2$ .

**Proposition 1.** Assume that  $0 < a < \frac{1}{2}$  (equivalently  $\sigma > 2$ ). Let  $u \in X^{\kappa, b}$  and  $v \in X_a^{s, b}$  with  $1/2 < b < 3/4$ ,  $1/4 < d < 1/2$  and  $|\kappa| - s \leq 1$ , so we have the bilinear estimate

$$(3.8) \quad \|\bar{u} \cdot v\|_{X^{\kappa, -d}} \leq C \|u\|_{X^{\kappa, b}} \cdot \|v\|_{X^{s, b}}.$$

The second bilinear estimate says that

**Proposition 2.** Assume that  $0 < a < \frac{1}{2}$  (equivalently  $\sigma > 2$ ). Let  $u, \tilde{u} \in X^{\kappa, b}$  with  $1/2 < b < 3/4$ ,  $1/4 < d < 1/2$  and  $s < \kappa + 1$  if  $\kappa \geq 0$  and  $s < 2\kappa + 1$  if  $\kappa < 0$  so it is valid the following estimate

$$(3.9) \quad \|u \cdot \tilde{u}\|_{X_a^{s, -d}} \leq C \|u\|_{X^{\kappa, b}} \cdot \|\tilde{u}\|_{X^{\kappa, b}}.$$

From lemmas 3 and 4, to prove the propositions above we only need to estimate the functionals  $J_i$  with  $i \in \{1, 2, \dots, 6\}$

*Proof of the Proposition 1.* We start by discussing the dispersion of relations. Note that

$$\begin{aligned}
|\omega - \omega_1 - \omega_2| &= |\xi^2 + \xi_1^2 - a\xi_2^2| \\
&\geq |1 - a|(\xi^2 + \xi_1^2) - 2a|\xi\xi_1|, \quad \text{suppose } 0 < a < \frac{1}{2} \\
&\geq (1 - a)(\xi^2 + \xi_1^2) - a(\xi^2 + \xi_1^2) = (1 - 2a)(\xi^2 + \xi_1^2).
\end{aligned}$$

So,

$$3 \max\{|\omega|, |\omega_1|, |\omega_2|\} \geq (1 - 2a) \max\{\xi^2, \xi_1^2\} \geq \frac{1 - 2a}{4} \xi_2^2.$$

Suppose  $|\xi_2| \geq 1$ , then we have

$$\frac{1}{\max\{|\omega|, |\omega_1|, |\omega_2|\}} \leq \frac{c}{|\xi_2|^2}.$$

Now, we define  $\mathcal{R}_i$ .

$$(3.10) \quad \mathcal{R}_1 = \left\{ |\xi_2| \geq 1, |\omega| = \max\{|\omega|, |\omega_1|, |\omega_2|\} \right\} \cup \left\{ |\xi_2| \leq 1 \right\} \subset \mathbb{R}_{\xi, \tau, \xi_2, \tau_2}^4;$$

$$(3.11) \quad \mathcal{R}_2 = \left\{ |\xi_2| \geq 1, |\omega_1| = \max\{|\omega|, |\omega_1|, |\omega_2|\} \right\} \subset \mathbb{R}_{\xi, \tau, \xi_2, \tau_2}^4;$$

$$(3.12) \quad \mathcal{R}_3 = \left\{ |\xi_2| \geq 1, |\tau_2 + a\xi^2| = \max\{|\omega|, |\omega_1|, |\omega_2|\} \right\} \subset \mathbb{R}_{\xi, \tau, \xi_2, \tau_2}^4.$$

Let us show that (3.3) is bounded. Indeed, if  $|\xi_2| \leq 1$  then (3.3) is equivalent to

$$\frac{1}{\langle \omega \rangle^{2d}} \int_{|\xi_2| \leq 1} \frac{1}{\langle \tau - (a-1)\xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 \leq c.$$

If  $|\xi_2| \geq 1$  then (3.3) is bounded by

$$\int_{|\xi_2| \geq 1} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|+4d} \chi_{\mathcal{R}_1}}{\langle \tau - (a-1)\xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2.$$

Now if  $-s + |\kappa| + 2d \leq 0$  then (3.3) is bounded, that is, enough that  $|\kappa| - s \leq 2d < 1$  because  $b > 1/2$ .

To prove that (3.4) is bounded, it is sufficient to note that the integral below is higher than (3.4) and that converge since  $|\kappa| - s \leq 2b$  and that  $2d > 1/2$ , that is,  $b < 3/4$ .

$$\int_{\mathbb{R}} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|-4b} \chi_{\mathcal{R}_2}}{\langle \tau_2 + 2\xi^2 + \xi_2^2 - 2\xi\xi_2 \rangle^{2d}} d\xi.$$

Analogously we prove that (3.5) is bounded, supposing that  $|\kappa| - s \leq 2b$  and  $b < 3/4$ . □

It now remains to prove the bounded of the second non-linear term of the system.

*Proof of the Proposition 2.* Note that

$$\begin{aligned}
|\lambda - \lambda_1 - \lambda_2| &= |a\xi^2 - \xi_1^2 - \xi_2^2| \\
&\geq |1 - a|(\xi_1^2 + \xi_2^2) - 2a|\xi_1\xi_2|, \quad \text{suppose } 0 < a < \frac{1}{2} \\
&\geq (1 - a)(\xi_1^2 + \xi_2^2) - a(\xi_1^2 + \xi_2^2) = (1 - 2a)(\xi_1^2 + \xi_2^2).
\end{aligned}$$

indeed  $\xi = \xi_1 + \xi_2$ , such that  $|\xi| \leq |\xi_1| + |\xi_2| \leq 2 \max\{\xi_1, \xi_2\}$ . Hence,

$$3 \max\{|\lambda|, |\lambda_1|, |\lambda_2|\} \geq (1 - 2a) \max\{\xi_1^2, \xi_2^2\} \geq \frac{1 - 2a}{4} \xi^2.$$

Thereby, supposing that  $|\xi| \geq 1$ , we have

$$\frac{1}{\max\{|\lambda|, |\lambda_1|, |\lambda_2|\}} \leq \frac{c}{|\xi|^2}.$$

Now, we define the regions  $\mathcal{S}_i$ .

$$(3.13) \quad \mathcal{S}_1 = \left\{ |\xi| \geq 1, |\lambda| = \max\{|\lambda|, |\lambda_1|, |\lambda_2|\} \right\} \cup \left\{ |\xi| \leq 1 \right\} \subset \mathbb{R}_{\xi, \tau, \xi_2, \tau_2}^4,$$

$$(3.14) \quad \mathcal{S}_2 = \left\{ |\xi| \geq 1, |\lambda_1| = \max\{|\lambda|, |\lambda_1|, |\lambda_2|\} \right\} \subset \mathbb{R}_{\xi, \tau, \xi_2, \tau_2}^4,$$

$$(3.15) \quad \mathcal{S}_3 = \left\{ |\xi| \geq 1, |\tau_2 + \xi^2| = \max\{|\lambda|, |\lambda_1|, |\lambda_2|\} \right\} \subset \mathbb{R}_{\xi, \tau, \xi_2, \tau_2}^4.$$

For  $\kappa \geq 0$ , we have  $\langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \leq \langle \xi \rangle^{-2\kappa}$  and in this case

$$(3.16) \quad J_4 \leq \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa+4d} \chi_{\mathcal{S}_1}}{\langle \tau + \xi^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2.$$

Thereby,  $J_4$  is bounded since  $s - \kappa + 2d < 0$  for  $s - \kappa < 2d$ .

Note that  $J_5$  and  $J_6$  are equivalent to,

$$(3.17) \quad J_5 \leq \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa-4b} \chi_{\mathcal{S}_2}}{\langle \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2 \rangle^{2d}} d\xi$$

and

$$(3.18) \quad J_6 \leq \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa-4b} \chi_{\mathcal{S}_3}}{\langle \tau_1 + a\xi^2 + \xi_2^2 \rangle^{2d}} d\xi_2,$$

and that they are bounded since  $s - \kappa < 2b$  and  $2d > \frac{1}{2}$ , this is,  $b < \frac{3}{4}$ .

The case  $\kappa < 0$  we need to separate in sub-cases as follows:

1. Supposing  $|\xi_1| \leq \frac{2}{3}|\xi_2|$ : then  $\langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \leq \langle \xi_2 \rangle^{-4\kappa}$ . Moreover,  $|\xi_2| \leq |\xi_1| + |\xi| \leq \frac{2|\xi_2|}{3} + |\xi|$ , hence  $|\xi_2| \leq 3|\xi|$ . Therefore,

$$\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \leq \langle \xi \rangle^{2s-4\kappa}.$$

2. Supposing  $|\xi_2| \leq \frac{2}{3}|\xi_1|$ , the following results, that is,

$$\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \leq \langle \xi \rangle^{2s-4\kappa}.$$

3. Missing case,  $\frac{2}{3}|\xi_2| < |\xi_1| < \frac{3}{2}|\xi_2|$ .

- (a) If  $\xi_1, \xi_2 \geq 0$  then  $\frac{2}{3}\xi_2 < \xi_1 < \frac{3}{2}\xi_2 \implies \frac{5}{3}\xi_2 < \xi < \frac{5}{2}\xi_2$ . Hence,

$$\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \leq \langle \xi \rangle^{2s-4\kappa}.$$

- (b) If  $\xi_1, \xi_2 \leq 0$  then  $-\frac{2}{3}\xi_2 < -\xi_1 < -\frac{3}{2}\xi_2 \implies -\frac{5}{3}\xi_2 < -\xi < -\frac{5}{2}\xi_2$ ,

thereby  $|\xi_2| < \frac{3}{5}|\xi|$ . Hence,

$$\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \leq \langle \xi \rangle^{2s-4\kappa}.$$

- (c) If  $\xi_1 > 0$  and  $\xi_2 < 0$  then  $-\frac{2}{3}\xi_2 < \xi_1 < \frac{3}{2}\xi_2 \implies \frac{1}{3}\xi_2 < \xi < \frac{1}{2}\xi_2$ ,  
thereby  $|\xi| < \frac{1}{2}|\xi_2|$ .

- (d) If  $\xi_1 < 0$  and  $\xi_2 > 0$  then  $\frac{2}{3}\xi_2 < -\xi_1 < \frac{3}{2}\xi_2 \implies \frac{-1}{3}\xi_2 < -\xi < \frac{1}{2}\xi_2$ ,  
thereby  $|\xi| < \frac{1}{2}|\xi_2|$ .

The cases (1), (2), (3.a) and (3.b) are true for  $\kappa < 0$  and  $s < 2\kappa + 1$ .

Indeed, given  $\mathcal{A} \subset \mathbb{R}^4$  the set of the elements of the  $\mathbb{R}^4$  that satisfies one of the conditions (1), (2), (3.a) or (3.b), given  $\mathcal{B} = \mathbb{R}^4 \setminus \mathcal{A}$ . Now consider  $\mathcal{A}_i = \mathcal{S}_i \cap \mathcal{A}$  and  $\mathcal{B}_i = \mathcal{S}_i \cap \mathcal{B}$ .

Analyzing the restrictions  $\mathcal{A}_i$ , we get:

$$\begin{aligned} J_4 &= \frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{B}_1}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 \\ &\leq \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-4\kappa-4d} \chi_{\mathcal{A}_1}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2. \end{aligned}$$

Then  $J_4$  is bounded for  $s \leq 2\kappa + 2d$  and  $b < 3/4$ .

$$\begin{aligned} J_5 &= \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{A}_2}}{\langle \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2 \rangle^{2d}} d\xi \\ &\leq \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-4\kappa-4b} \chi_{\mathcal{A}_2}}{\langle \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2 \rangle^{2d}} d\xi \end{aligned}$$

and  $J_5$  is bounded for  $s \leq 2\kappa + 2b$  and  $1/2 < b$ .

$$\begin{aligned} J_6 &= \frac{1}{\langle \lambda_1 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{A}_3}}{\langle \tau_1 + a\xi^2 + \xi_2^2 \rangle^{2d}} d\xi_2 \\ &\leq \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-4\kappa-4b} \chi_{\mathcal{A}_3}}{\langle \tau_1 + a\xi^2 + \xi_2^2 \rangle^{2d}} d\xi_2. \end{aligned}$$

Then  $J_6$  is also bounded for  $s \leq 2\kappa + 2b$  and  $1/2 < b$ .

To analyze the remaining cases (which boil down to take  $|\xi| < \frac{1}{2}|\xi_2|$  and  $|\xi_1| \sim |\xi_2|$ ) let consider then as regions  $\mathcal{B}_i$ :

We start by estimating  $J_4$ .

$$\begin{aligned} J_4 &= \frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{B}_1}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 \\ &\leq \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-4d} \langle \xi_1 \rangle^{-4\kappa} \chi_{\mathcal{B}_1}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 \\ &\leq \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-4d} \langle \xi_1 \rangle^{-4\kappa} \chi_{\mathcal{B}_1}}{2|\xi_2 - \xi| \langle \eta \rangle^{2b}} d\eta \end{aligned}$$

Now,  $|\xi_2 - \xi| \geq |\xi_2| - |\xi| \geq \frac{1}{2}|\xi_2| \sim \frac{1}{2}|\xi_1|$ .

Hence,  $J_4 \leq \langle \xi \rangle^{2s-4d} \langle \xi_1 \rangle^{-4\kappa-1} \int_{\mathbb{R}} \frac{d\eta}{\langle \eta \rangle^{2b}}$ , that is bounded because  $2b > 1$  and  $2s \leq 4\kappa + 2$ .

$$\langle \xi \rangle^{2s-4d} \langle \xi_1 \rangle^{-4\kappa-1} \leq \langle \xi \rangle^{2s-4\kappa-4d-1}.$$

We continue to estimate  $J_5$ :

$$\begin{aligned} J_5 &= \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{B}_2}}{\langle \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2 \rangle^{2d}} d\xi \\ &\leq \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_2 \rangle^{-4\kappa} \chi_{\mathcal{B}_2}}{\langle \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2 \rangle^{2d}} d\xi \end{aligned}$$

Setting  $\eta = \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2$ , such that  $d\eta = 2(\xi_2 + (a-1)\xi)d\xi$ . Now, as  $0 < a < \frac{1}{2}$ , it follows  $|a-1| < 1$  and thereby  $|\xi_2 + (a-1)\xi| \geq \frac{1}{2}|\xi_2|$ . Observe still that

$$\begin{aligned} |\eta| &= |\tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2| \\ &= |\lambda_2 + ((a-1)\xi^2 - 2\xi_2^2 + 2\xi\xi_2)| \\ &\leq |\lambda_2| + |(a-1)\xi^2 - 2\xi_2^2 + 2\xi\xi_2| \leq |\tau_2 + \xi_2| + 4|\xi_2|^2 \\ &\leq c|\lambda_2|. \end{aligned}$$

Thereby,

$$\begin{aligned}
J_5 &\leq \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\langle \eta \rangle \leq c \langle \lambda_2 \rangle} \frac{\langle \xi \rangle^{2s} \langle \xi_2 \rangle^{-4\kappa-1}}{\langle \eta \rangle^{2d}} d\eta \\
&\leq \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\langle \eta \rangle \leq c \langle \lambda_2 \rangle} \frac{\langle \xi_2 \rangle^{\max\{0, 2s\}-4\kappa-1}}{\langle \eta \rangle^{2d}} d\eta, \quad \text{because } |\xi| < \frac{1}{2} |\xi_2| \\
&\leq \langle \xi_2 \rangle^{\max\{0, 2s\}-4\kappa-1} \frac{\langle \lambda_2 \rangle^{2d}}{\langle \lambda_2 \rangle^{2b}} \\
&\leq \langle \xi_2 \rangle^{\max\{0, 2s\}-4\kappa-1} \langle \lambda_2 \rangle^{-2b+2d} \leq \langle \xi_2 \rangle^{\max\{0, 2s\}-4\kappa-1-2b+2d}.
\end{aligned}$$

We prove  $J_6$ . Remember that

$$\begin{aligned}
J_6 &= \frac{1}{\langle \lambda_1 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{B_3}}{\langle \tau_1 + a\xi^2 + \xi_2^2 \rangle^{2d}} d\xi_2 \\
&\leq \frac{1}{\langle \lambda_1 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-4\kappa} \chi_{B_3}}{\langle \tau_1 + a\xi^2 + \xi_2^2 \rangle^{2d}} d\xi_2.
\end{aligned}$$

Let  $\eta = \tau_1 + a\xi^2 + \xi_2^2$ , such that  $d\eta = 2\xi_2 d\xi_2$ . Now,

$$\begin{aligned}
|\eta| &= |\tau_1 + a\xi^2 + \xi_2^2| \\
&= |(\lambda_1) + (a\xi^2 + \xi_2^2 - \xi_1^2)| \\
&\leq c|\lambda_1|.
\end{aligned}$$

We use the fact that  $|\xi_1| \sim |\xi_2|$ , it follows

$$\begin{aligned}
J_6 &\leq \frac{1}{\langle \lambda_1 \rangle^{2b}} \int_{\langle \eta \rangle \leq c \langle \lambda_1 \rangle} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-4\kappa}}{|\xi_1| \langle \eta \rangle^{2d}} d\xi_2 \\
&\leq \langle \xi_1 \rangle^{\max\{0, 2s\}-4\kappa-1} \frac{\langle \lambda_1 \rangle^{2d}}{\langle \lambda_1 \rangle^{2b}} \\
&\leq \langle \xi_1 \rangle^{\max\{0, 2s\}-4\kappa-1-2b+2d}.
\end{aligned}$$

As  $1/2 < b < 3/4$  and  $1/4 < d < 1/2$  such that  $-1 < -2b + 2d < 0$  and hence we can take  $b$  and  $d$  so that  $2s - 4\kappa - 1 - 2b + 2d < 0$  if  $s < 2\kappa + 1$ .

Thereby, we completed the proof of Proposition 2.  $\square$

**Remark 2.** The lines  $s = -\kappa - 1$  and  $s = 2\kappa + 1$  intersect at the point where  $\kappa = -\frac{2}{3}$ .

### 3.2 Bilinear estimates for $\sigma < 2$

For  $a > 1/2$  the following propositions are valid

**Proposition 3.** Assume that  $a > 1/2$  (equivalently  $\sigma < 2$ ). Let  $u \in X^{\kappa, b}$  and  $v \in X_a^{s, b}$ . Then the estimative bilinear below is valid if  $1/2 < b < 3/4$ ,  $1/4 < d < 1/2$  and  $|\kappa| - s \leq 1/2$ .

$$(3.19) \quad \|\overline{u} \cdot v\|_{X^{\kappa, -d}} \leq C \|u\|_{X^{\kappa, b}} \cdot \|v\|_{X^{s, b}}.$$

The second estimate tells us that

**Proposition 4.** *Assume that  $a > 1/2$  (equivalently  $\sigma < 2$ ). Let  $u, \tilde{u} \in X^{\kappa, b}$  with  $1/2 < b < 3/4$  and  $1/4 < d < 1/2$ .*

$$(3.20) \quad \|u \cdot \tilde{u}\|_{X^{s, -d}} \leq C \|u\|_{X^{\kappa, b}} \cdot \|\tilde{u}\|_{X^{\kappa, b}}.$$

For  $s \leq \min \{\kappa + 1/2, 2\kappa + 1/2\}$ .

*Proof of Proposition 3:* Similarly the earlier we start by considering the dispersion relation.

Note that

$$\begin{aligned} |\omega - \omega_1 - \omega_2| &= |\xi^2 + \xi_1^2 - a\xi_2^2| \\ &\geq |2\xi^2 - 2\xi\xi_2 + (1-a)\xi_2^2|, \quad \text{using } a > \frac{1}{2} \text{ have} \\ &= 2|\xi - \mu_a\xi_2| \cdot |\xi - (1-\mu_a)\xi_2|, \quad \text{where } \mu_a = \frac{1 - \sqrt{2a-1}}{2}. \end{aligned}$$

Note that dispersion relation above has two areas of uniqueness, the lines  $\xi = \mu_a\xi_2$  and  $\xi = (1-\mu_a)\xi_2$  making it difficult to use the relationship. Observe that if  $a = \frac{1}{2}$  then  $\mu_a = 1 - \mu_a = \frac{1}{2}$  and if  $a = 1$  then  $\mu_a = 0$  (the case  $a = \frac{1}{2}$  it will be treated separately, already the case  $a = 1$  does not require much attention despite being the case without modification).

Before, consider

$$\begin{aligned} \mathcal{A}_1 &= \{|\xi_2| \leq 1\} \subset \mathbb{R}^4, \\ \mathcal{A}_2 &= \left\{ |\xi_2| \geq 1, |(1-a)\xi_2 - \xi| > \frac{2a-1}{4}|\xi_2| \right\} \subset \mathbb{R}^4, \\ \mathcal{A}_3 &= \left\{ |\xi_2| \geq 1, \left| \xi - \frac{1}{2}\xi_2 \right| > \frac{2a-1}{4}|\xi_2| \right\} \subset \mathbb{R}^4. \end{aligned}$$

Note that if  $\left| \xi - \frac{1}{2}\xi_2 \right| \leq \frac{2a-1}{4}|\xi_2|$  and  $\left| \xi - \frac{1}{2}\xi_2 \right| \leq \frac{2a-1}{4}|\xi_2|$  and still  $|\xi_2| \geq 1$  then

$$\begin{aligned} \left( a - \frac{1}{2} \right) |\xi_2| &= \left| \left( \xi - \frac{1}{2}\xi_2 \right) + ((1-a)\xi_2 - \xi) \right| \\ &\leq \frac{2a-1}{4}|\xi_2| + \frac{2a-1}{4}|\xi_2| = \frac{1}{2} \left( a - \frac{1}{2} \right) |\xi_2|. \end{aligned}$$

Absurd, that is,  $\mathbb{R}^4 = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ .  
Now consider,

$$\begin{aligned}\mathcal{A}_{3,1} &= \mathcal{A}_3 \cap \{|\omega| \geq \max\{|\omega_1|, |\omega_2|\}\}, \\ \mathcal{A}_{3,2} &= \mathcal{A}_3 \cap \{|\omega_2| \geq \max\{|\omega_1|, |\omega|\}\}, \\ \mathcal{A}_{3,3} &= \mathcal{A}_3 \cap \{|\omega_1| \geq \max\{|\omega|, |\omega_2|\}\}.\end{aligned}$$

Remember that  $|2\xi^2 + \xi_2^2 - 2\xi\xi_2| \leq 3 \max\{|\omega|, |\omega_1|, |\omega_2|\}$ .

Now, we define the regions  $\mathcal{R}_i$  (see Lemma 3). Let  $\mathcal{R}_1 = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_{3,1}$ ,  $\mathcal{R}_2 = \mathcal{A}_{3,2}$  and  $\mathcal{R}_3 = \mathcal{A}_{3,3}$ .

We will show that (3.3) is bounded. Indeed, if  $|\xi_2| \leq 1$  then (3.3) is equivalent to

$$\frac{1}{\langle \omega \rangle^{2d}} \int_{|\xi_2| \leq 1} \frac{1}{\langle \tau - (a-1)\xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 \leq c.$$

If  $|\xi_2| \geq 1$  then (3.3) is bounded for

$$\frac{1}{\langle \omega \rangle^{2d}} \int_{|\xi_2| \geq 1} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|} \chi_{\mathcal{A}_2}}{\langle \tau - (a-1)\xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2.$$

Making the change of variable  $\eta = \tau - (a-1)\xi_2^2 - 2\xi\xi_2 + \xi^2$ , such that

$$d\eta = -2((1-a)\xi_2 - \xi) d\xi_2$$

and, using the fact that  $|\kappa| - s \leq 1/2$ , we get thereby that

$$\begin{aligned}\frac{1}{\langle \omega \rangle^{2d}} \int_{|\xi_2| \geq 1} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|} \chi_{\mathcal{A}_2}}{\langle \tau - (a-1)\xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 &\leq c \frac{1}{\langle \omega \rangle^{2d}} \int_{|\xi_2| \geq 1} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|-1} \chi_{\mathcal{A}_2}}{\langle \eta \rangle^{2b}} d\eta \\ &\leq c \frac{1}{\langle \omega \rangle^{2d}} \int_{\mathbb{R}} \frac{1}{\langle \eta \rangle^{2b}} d\eta \leq c.\end{aligned}$$

Observe now that in  $\mathcal{A}_{3,1}$  we have:

$$|(1-a)\xi_2 - \xi| = \left| \frac{1}{2}\xi_2 - \xi + \left( \frac{1}{2} - a \right) \xi_2 \right| \geq \left( a - \frac{1}{2} \right) |\xi_2| - \frac{2a-1}{4} |\xi_2| \geq c|\xi_2|.$$

To complete the bounded (3.3) just estimate

$$\begin{aligned}\frac{1}{\langle \omega \rangle^{2d}} \int_{|\xi_2| \geq 1} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|} \chi_{\mathcal{A}_{3,1}}}{\langle \tau - (a-1)\xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 &\leq c \frac{1}{\langle \omega \rangle^{2d}} \int_{|\xi_2| \geq 1} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|-1} \chi_{\mathcal{A}_{3,1}}}{\langle \eta \rangle^{2b}} d\eta \\ &\leq c \frac{1}{\langle \omega \rangle^{2d}} \int_{\mathbb{R}} \frac{1}{\langle \eta \rangle^{2b}} d\eta \leq c.\end{aligned}$$

To prove that (3.4) is bounded just observe that

$$\begin{aligned}\frac{1}{\langle \tau + a\xi^2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|} \chi_{\mathcal{R}_2}}{\langle \tau_2 + 2\xi^2 + \xi_2^2 - 2\xi\xi_2 \rangle^{2d}} d\xi &= \frac{1}{\langle \omega_2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|} \chi_{\mathcal{A}_{3,2}}}{\langle \tau_2 + 2\xi^2 + \xi_2^2 - 2\xi\xi_2 \rangle^{2d}} d\xi \\ &\leq \frac{1}{\langle \omega_2 \rangle^{2b}} \int_{\langle \eta \rangle \leq 4\langle \omega_2 \rangle} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|-1}}{\langle \eta \rangle^{2d}} d\eta \\ &\leq \frac{1}{\langle \omega_2 \rangle^{2b-2d}}.\end{aligned}$$



In the first inequality above, we make the change of variable  $\eta = \tau_2 + 2\xi^2 + \xi_2^2 - 2\xi\xi_2$  and use the fact that

$$|\eta| = |\omega_2 + (\omega - \omega_1 - \omega_2)| \leq 4|\omega_2|.$$

It remains to show (3.5). Analogously, the last estimate we get is

$$\begin{aligned} \frac{1}{\langle \tau - \xi_1^2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|} \chi_{\mathcal{R}_3}}{\langle \tau_1 - a\xi_2^2 + \xi^2 \rangle^{2d}} d\xi_2 &= \frac{1}{\langle \omega_1 \rangle^{2b}} \int_{|\xi_2| > 1} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|} \chi_{\mathcal{A}_{3,3}}}{\langle \tau_1 - a\xi_2^2 + \xi^2 \rangle^{2d}} d\xi_2 \\ &\leq \frac{1}{\langle \omega_1 \rangle^{2b}} \int_{\langle \eta \rangle \leq 4\langle \omega_1 \rangle} \frac{\langle \xi_2 \rangle^{-2s+2|\kappa|-1}}{\langle \eta \rangle^{2d}} d\eta \\ &\leq \frac{1}{\langle \omega_1 \rangle^{2b-2d}}. \end{aligned}$$

Observe that use the fact that  $\tau_1 - a\xi_2^2 + \xi^2 = \omega_1 + (\omega - \omega_1 - \omega_2)$ .

This ends the proof of the first inequality.  $\square$

*Proof of Proposition 4.* To check the validity of the second estimate we started observing that:

$$\begin{aligned} |\lambda - \lambda_1 - \lambda_2| &= |a\xi^2 - \xi_1^2 - \xi_2^2| \\ &\geq |2\xi_2^2 - 2\xi\xi_2 + (1-a)\xi^2| \quad \text{using } a > \frac{1}{2} \text{ we have} \\ &= 2|\xi_2 - \mu_a \xi| \cdot |\xi_2 - (1-\mu_a)\xi|, \quad \text{where } \mu_a = \frac{1 - \sqrt{2a-1}}{2}. \end{aligned}$$

The dispersion relation above is zero in two straight.

Now defining,

$$\begin{aligned} \mathcal{B}_1 &= \{|\xi| \leq 1\} \subset \mathbb{R}^4, \\ \mathcal{B}_2 &= \left\{ |\xi| \geq 1, \left| \xi_2 - \frac{1}{2}\xi \right| > \frac{2a-1}{4}|\xi| \right\} \subset \mathbb{R}^4, \\ \mathcal{B}_3 &= \left\{ |\xi| \geq 1, |(1-a)\xi - \xi_2| > \frac{2a-1}{4}|\xi| \right\} \subset \mathbb{R}^4. \end{aligned}$$

Note that if  $\left| \xi_2 - \frac{1}{2}\xi \right| \leq \frac{2a-1}{4}|\xi|$  and  $\left| \xi_2 - \frac{1}{2}\xi \right| \leq \frac{2a-1}{4}|\xi|$  and still  $|\xi| > 1$  then

$$\begin{aligned} \left( a - \frac{1}{2} \right) |\xi| &= \left| \left( \xi_2 - \frac{1}{2}\xi \right) + ((1-a)\xi - \xi_2) \right| \\ &\leq \frac{2a-1}{4}|\xi| + \frac{2a-1}{4}|\xi| = \frac{1}{2} \left( a - \frac{1}{2} \right) |\xi|. \end{aligned}$$

Absurd, that is,  $\mathbb{R}^4 = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ .

Now consider

$$\begin{aligned}\mathcal{B}_{3,1} &= \mathcal{B}_3 \cap \{|\lambda| \geq \max\{|\lambda_1|, |\lambda_2|\}\}, \\ \mathcal{B}_{3,2} &= \mathcal{B}_3 \cap \{|\lambda_2| \geq \max\{|\lambda_1|, |\lambda|\}\}, \\ \mathcal{B}_{3,3} &= \mathcal{B}_3 \cap \{|\lambda_1| \geq \max\{|\lambda|, |\lambda_2|\}\}.\end{aligned}$$

we define the regions  $\mathcal{S}_i$  (see Lemma 4). Let  $\mathcal{S}_1 = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_{3,1}$ ,  $\mathcal{S}_2 = \mathcal{B}_{3,2}$  and  $\mathcal{S}_3 = \mathcal{B}_{3,3}$ .

For  $\kappa \geq 0$ , we have  $\langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \leq \langle \xi \rangle^{-2\kappa}$ .

$$(3.21) \quad J_4 \leq \frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa} \chi_{\mathcal{S}_1}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2,$$

$$(3.22) \quad J_5 \leq \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa} \chi_{\mathcal{S}_2}}{\langle \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2 \rangle^{2d}} d\xi,$$

$$(3.23) \quad J_6 \leq \frac{1}{\langle \lambda_1 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa} \chi_{\mathcal{S}_3}}{\langle \tau_1 + a\xi^2 + \xi_2^2 \rangle^{2d}} d\xi_2.$$

To complete that  $J_4$  is bounded just observe that (3.21)

$$\frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa} \chi_{\mathcal{B}_1}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 \leq \frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{1}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 \leq c;$$

$$\frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa} \chi_{\mathcal{B}_2}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 \leq \frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa-1}}{\langle \eta \rangle^{2b}} d\xi_2 \leq c \text{ and}$$

$$\frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa} \chi_{\mathcal{B}_{3,1}}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 \leq \frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa-1}}{\langle \eta \rangle^{2b}} d\xi_2 \leq c.$$

For the above estimates, it is important to use the fact  $b > 1/2$  and for the last inequality we use the fact that

$$\begin{aligned}\left| \xi_2 - \frac{1}{2}\xi \right| &= \left| (1-a)\xi - \xi_2 + \left(a - \frac{1}{2}\right)\xi \right| \\ &\geq \left(a - \frac{1}{2}\right)|\xi| - |(1-a)\xi - \xi_2| \\ &\geq \left(a - \frac{1}{2}\right)|\xi| - \frac{1}{2}\left(a - \frac{1}{2}\right)|\xi| = \frac{2a-1}{4}|\xi|.\end{aligned}$$

Let's estimate (3.22) using the fact that

$$\eta = \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2 = \lambda_2 + (\lambda - \lambda_1 - \lambda_2)$$

and  $d\eta = 2((1-a)\xi - \xi_2)d\xi$ , we have thereby

$$\begin{aligned} \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa} \chi_{\mathcal{B}_{3,2}}}{\langle \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2 \rangle^{2d}} d\xi &\leq \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\langle \eta \rangle \leq 4\langle \lambda_2 \rangle} \frac{\langle \xi \rangle^{2s-2\kappa-1}}{\langle \eta \rangle^{2d}} d\eta \\ &\leq \frac{1}{\langle \lambda_2 \rangle^{2b-2d}} \leq c. \end{aligned}$$

Now let estimate (3.23), we will do this analogous to previous

$$\begin{aligned} \frac{1}{\langle \lambda_1 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa} \chi_{\mathcal{S}_3}}{\langle \tau_1 + a\xi^2 + \xi_2^2 \rangle^{2d}} d\xi_2 &\leq \frac{1}{\langle \lambda_1 \rangle^{2b}} \int_{\langle \eta \rangle \leq 4\langle \lambda_1 \rangle} \frac{\langle \xi \rangle^{2s-2\kappa-1}}{\langle \eta \rangle^{2d}} d\eta \\ &\leq \frac{1}{\langle \lambda_1 \rangle^{2b-2d}} \leq c. \end{aligned}$$

This closes the case  $\kappa \geq 0$ .

The case  $\kappa < 0$  will be separated into sub-cases:

1. Supposing  $|\xi_1| \leq \frac{2}{3}|\xi_2|$ , then,  $\langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \leq \langle \xi_2 \rangle^{-4\kappa}$ . Moreover,  $|\xi_2| \leq |\xi_1| + |\xi| \leq \frac{2|\xi_2|}{3} + |\xi|$ , hence  $|\xi_2| \leq 3|\xi|$ . Therefore,

$$\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \leq \langle \xi \rangle^{2s-4\kappa}.$$

2. Supposing  $|\xi_2| \leq \frac{2}{3}|\xi_1|$  we have the same result, that is,

$$\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \leq \langle \xi \rangle^{2s-4\kappa}.$$

3. Just the case,  $\frac{2}{3}|\xi_2| < |\xi_1| < \frac{3}{2}|\xi_2|$ .

- (a) If  $\xi_1, \xi_2 \geq 0$  then  $\frac{2}{3}\xi_2 < \xi_1 < \frac{3}{2}\xi_2 \implies \frac{5}{3}\xi_2 < \xi < \frac{5}{2}\xi_2$ . Hence,

$$\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \leq \langle \xi \rangle^{2s-4\kappa}.$$

- (b) If  $\xi_1, \xi_2 \leq 0$  then  $\frac{-2}{3}\xi_2 < -\xi_1 < \frac{-3}{2}\xi_2 \implies \frac{-5}{3}\xi_2 < -\xi < \frac{-5}{2}\xi_2$ ,  
thereby  $|\xi_2| < \frac{3}{5}|\xi|$ . Hence,

$$\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \leq \langle \xi \rangle^{2s-4\kappa}.$$

- (c) If  $\xi_1 > 0$  and  $\xi_2 < 0$  then  $\frac{-2}{3}\xi_2 < \xi_1 < \frac{-3}{2}\xi_2 \implies \frac{1}{3}\xi_2 < \xi < \frac{-1}{2}\xi_2$ ,  
thereby  $|\xi| < \frac{1}{2}|\xi_2|$ .

(d) If  $\xi_1 < 0$  and  $\xi_2 > 0$  then  $\frac{2}{3}\xi_2 < -\xi_1 < \frac{3}{2}\xi_2 \implies \frac{-1}{3}\xi_2 < -\xi < \frac{1}{2}\xi_2$ ,  
thereby  $|\xi| < \frac{1}{2}|\xi_2|$ .

The cases (1), (2), (3.a) and (3.b) are valid for  $\kappa < 0$  and  $s < 2\kappa + \frac{1}{2}$ .

Indeed, let  $\mathcal{C} \subset \mathbb{R}^4$  the set of element  $\mathbb{R}^4$  that satisfies one of the conditions (1), (2), (3.a) or (3.b). Now consider  $\mathcal{C}_i = \mathcal{S}_i \cap \mathcal{C}$ .

Analyzing restrictions on  $\mathcal{C}_i$ , we get:

$$\begin{aligned} \frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{C}_1}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 &\leq \frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-4\kappa} \chi_{\mathcal{C}_1}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 \\ &\leq \frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-4\kappa-1}}{\langle \eta \rangle^{2b}} d\xi_2 \\ &\leq c, \text{ because } 1/2 < b < 1 \text{ and } s < 2\kappa + 1/2. \end{aligned}$$

$$\begin{aligned} \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{C}_2}}{\langle \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2 \rangle^{2d}} d\xi &\leq \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\langle \eta \rangle \leq 4\langle \lambda_2 \rangle} \frac{\langle \xi \rangle^{2s-4\kappa-1}}{\langle \eta \rangle^{2d}} d\eta \\ &\leq \frac{1}{\langle \lambda_2 \rangle^{2b-2d}} \leq c. \end{aligned}$$

$$\begin{aligned} \frac{1}{\langle \lambda_1 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{C}_3}}{\langle \tau_1 + a\xi^2 + \xi_2^2 \rangle^{2d}} d\xi_2 &\leq \frac{1}{\langle \lambda_1 \rangle^{2b}} \int_{\langle \eta \rangle \leq 4\langle \lambda_1 \rangle} \frac{\langle \xi \rangle^{2s-4\kappa-1}}{\langle \eta \rangle^{2d}} d\eta \\ &\leq \frac{1}{\langle \lambda_1 \rangle^{2b-2d}} \leq c. \end{aligned}$$

Consider  $\mathcal{D} = \mathbb{R}^4 \setminus \mathcal{C}$  and  $\mathcal{D}_i = \mathcal{S}_i \cap \mathcal{D}$ . To examine the other cases (which boil down to take  $|\xi| < \frac{1}{2}|\xi_2|$  and  $|\xi_1| \sim |\xi_2|$ ) let's consider the regions  $\mathcal{D}_i$ :

We begin by estimating  $J_4$ .

$$\begin{aligned} \frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{D}_1}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 &\leq \frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-4\kappa} \chi_{\mathcal{D}_1}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 \\ &\leq \frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa}}{\langle \eta \rangle^{2b}} d\eta. \end{aligned}$$

Now,  $|\xi_2 - \xi| \geq |\xi_2| - |\xi| \geq \frac{1}{2}|\xi_2| \sim \frac{1}{2}|\xi_1|$ .

Hence,  $J_4 \leq \langle \xi \rangle^{2s-4d} \langle \xi_1 \rangle^{-4\kappa-1} \int_{\mathbb{R}} \frac{d\eta}{\langle \eta \rangle^{2b}}$ , that is bounded because  $2b > 1$  and  $2s \leq 4\kappa + 2$ .

$$\langle \xi \rangle^{2s-4d} \langle \xi_1 \rangle^{-4\kappa-1} \leq \langle \xi \rangle^{2s-4\kappa-1-4d} \leq \langle \xi \rangle^{1-4d}.$$

Estimate  $J_5$ .

$$\begin{aligned} J_5 &= \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{B}_2}}{\langle \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2 \rangle^{2d}} d\xi \\ &\leq \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_2 \rangle^{-4\kappa} \chi_{\mathcal{B}_2}}{\langle \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2 \rangle^{2d}} d\xi. \end{aligned}$$

Setting  $\eta = \tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2$ , such that  $d\eta = 2(\xi_2 + (a-1)\xi)d\xi$ . Now, as  $0 < a < \frac{1}{2}$ , we have  $|a-1| \leq 1$  and thereby  $|\xi_2 + (a-1)\xi| \geq \frac{1}{2}|\xi_2|$ . We still observe that

$$\begin{aligned} |\eta| &= |\tau_2 + (a-1)\xi^2 - \xi_2^2 + 2\xi\xi_2| \\ &= |(\lambda_2) + ((a-1)\xi^2 - 2\xi_2^2 + 2\xi\xi_2)| \\ &\leq |\lambda_2| + |(a-1)\xi^2 - 2\xi_2^2 + 2\xi\xi_2| \leq |\tau_2 + \xi_2| + 4|\xi_2|^2 \\ &\leq c|\lambda_2|. \end{aligned}$$

Thereby,

$$\begin{aligned} J_5 &\leq \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\langle \eta \rangle \leq c\langle \lambda_2 \rangle} \frac{\langle \xi \rangle^{2s} \langle \xi_2 \rangle^{-4\kappa-1}}{\langle \eta \rangle^{2d}} d\eta \\ &\leq \frac{1}{\langle \lambda_2 \rangle^{2b}} \int_{\langle \eta \rangle \leq c\langle \lambda_2 \rangle} \frac{\langle \xi_2 \rangle^{\max\{0, 2s\}-4\kappa-1}}{\langle \eta \rangle^{2d}} d\eta \quad \text{because } |\xi| < \frac{1}{2}|\xi_2| \\ &\leq \langle \xi_2 \rangle^{\max\{0, 2s\}-4\kappa-1} \frac{\langle \lambda_2 \rangle^{1-2d}}{\langle \lambda_2 \rangle^{2b}} \\ &\leq \langle \xi_2 \rangle^{\max\{0, 2s\}-4\kappa-1} \langle \lambda_2 \rangle^{1-2d-2b} \leq \langle \xi_2 \rangle^{\max\{0, 2s\}-4\kappa-2}. \end{aligned}$$

Since  $1-2b-2d < -1/2$ . Now, estimate  $J_6$ . Remember that

$$\begin{aligned} J_6 &= \frac{1}{\langle \lambda_1 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \chi_{\mathcal{B}_3}}{\langle \tau_1 + a\xi^2 + \xi_2^2 \rangle^{2d}} d\xi_2 \\ &\leq \frac{1}{\langle \lambda_1 \rangle^{2b}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-4\kappa} \chi_{\mathcal{B}_3}}{\langle \tau_1 + a\xi^2 + \xi_2^2 \rangle^{2d}} d\xi_2. \end{aligned}$$

Make  $\eta = \tau_1 + a\xi^2 + \xi_2^2$ , such that  $d\eta = 2\xi_2 d\xi_2$ . Now,

$$\begin{aligned} |\eta| &= |\tau_1 + a\xi^2 + \xi_2^2| \\ &= |(\lambda_1) + (a\xi^2 + \xi_2^2 - \xi_1^2)| \\ &\leq c|\lambda_1|. \end{aligned}$$

We use the fact that  $|\xi_1| \sim |\xi_2|$ , we have

$$\begin{aligned} J_6 &\leq \frac{1}{\langle \lambda_1 \rangle^{2b}} \int_{\langle \eta \rangle \leq c\langle \lambda_1 \rangle} \frac{\langle \xi \rangle^{2s} \langle \xi_1 \rangle^{-4\kappa}}{|\xi_1| \langle \eta \rangle^{2d}} d\xi_2 \\ &\leq \langle \xi_1 \rangle^{\max\{0, 2s\}-4\kappa-1} \frac{\langle \lambda_1 \rangle^{1-2d}}{\langle \lambda_1 \rangle^{2b}} \\ &\leq \langle \xi_1 \rangle^{\max\{0, 2s\}-4\kappa-2}. \end{aligned}$$

We completed the proof of Proposition 4.  $\square$

**Remark 3.** The lines  $s = -\kappa - 1/2$  and  $s = 2\kappa + 1/2$ . They intersect at the point satisfying  $\kappa = -\frac{1}{3}$ .

### 3.3 Bilinear estimates for $\sigma = 2$

Next we prove new bilinear estimates for the interaction terms in case  $\sigma = 2$

**Proposition 5.** Assume that  $a = 1/2$  (equivalently  $\sigma = 2$ ). If  $1/2 < b < 3/4$ ,  $1/4 < d < 1/2$  and  $|\kappa| \leq s$ . Then, for  $u \in X^{\kappa, b}$  and  $v \in X_a^{s, b}$ , the estimative below is true

$$(3.24) \quad \|\bar{u} \cdot v\|_{X^{\kappa, -d}} \leq C \|u\|_{X^{\kappa, b}} \cdot \|v\|_{X^{s, b}}.$$

The second bilinear estimate tells us that

**Proposition 6.** Assume that  $a = 1/2$  (equivalently  $\sigma = 2$ ). Let  $u, \tilde{u} \in X^{\kappa, b}$ . Then

$$(3.25) \quad \|u \cdot \tilde{u}\|_{X^{s, -d}} \leq C \|u\|_{X^{\kappa, b}} \cdot \|\tilde{u}\|_{X^{\kappa, b}}.$$

is valid if  $1/2 < b < 3/4$ ,  $1/4 < d < 1/2$  and  $0 \leq s \leq \kappa$ .

*Proof of Proposition 5:* As in other cases, we begin noting that

$$\begin{aligned} |\omega - \omega_1 - \omega_2| &= \left| \xi^2 + \xi_1^2 - \frac{1}{2}\xi_2^2 \right| \\ &= \left| 2\xi^2 + 2\xi\xi_2 + \frac{1}{2}\xi_2^2 \right| \\ &= 2 \left| \xi + \frac{1}{2}\xi_2 \right|^2. \end{aligned}$$

In this case, we do not have to take the dispersion relation.

In this case just consider  $\mathcal{R}_1 = \mathbb{R}^4$  and  $\mathcal{R}_2 = \mathcal{R}_3 = \emptyset$ . Thereby, it remains to show that  $J_1$  is bounded.

If  $|\kappa| \leq s$  then  $J_1$  (3.3) is equivalent to

$$\frac{1}{\langle \omega \rangle^{2d}} \int_{|\xi_2| \leq 1} \frac{1}{\langle \tau - \frac{1}{2}\xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2 \leq c,$$

because  $b > 1/2$  and  $d > 0$ .

This ends the proof of the proposition.  $\square$

*Proof of Proposition 6:* As in the previous, we cannot take advantage of the dispersion relation. So let us take  $\mathcal{S}_1 = \mathbb{R}^4$  and  $\mathcal{S}_2 = \mathcal{S}_3 = \emptyset$ .

Just estimate  $J_4$ , thereby, initially assume that  $\kappa \geq 0$  and thereby  $\langle \xi_1 \rangle^{-2\kappa} \langle \xi_2 \rangle^{-2\kappa} \leq \langle \xi \rangle^{-2\kappa}$ :

$$J_4 \leq \frac{1}{\langle \lambda \rangle^{2d}} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s-2\kappa} \chi_{\mathcal{S}_1}}{\langle \tau + \xi_2^2 - 2\xi\xi_2 + \xi^2 \rangle^{2b}} d\xi_2.$$

Assuming that  $s \leq \kappa$  and that  $b > 1/2$  and  $d > 0$  bounded  $J_4$ .  $\square$

## 4 Local existence for low regularity data

In this section we prove by Banach Fixed Point Theorem the result of local well-posedness.

We will demonstrate only the case  $0 < a < 1/2$  because the other cases follow the same arguments.

Let's consider the following functional space where we get our solution:

$$(4.1) \quad \Sigma_\mu := \left\{ (u, v) \in X^{\kappa, \frac{1}{2}+\mu} \times X_a^{s, \frac{1}{2}+\mu}; \|u\|_{X^{\kappa, \frac{1}{2}+\mu}} \leq M_1, \|v\|_{X_a^{s, \frac{1}{2}+\mu}} \leq M_2 \right\},$$

where  $0 < \mu \ll 1$  and  $M_1, M_2 > 0$  will be chosen below.

We note that  $\Sigma_\mu$  is a complete metric space with the standard:

$$\|(u, v)\|_{\Sigma_\mu} := \|u\|_{X^{\kappa, \frac{1}{2}+\mu}} + \|v\|_{X_a^{s, \frac{1}{2}+\mu}}.$$

For  $(u, v) \in \Sigma_\mu$ , we define the maps

$$(4.2) \quad \Phi_1(u, v) = \psi_1(t)U(t)u_0 - i\psi_T(t) \int_0^t U(t-t') \{ \theta u(t') - (\bar{u} \cdot v)(t') \} dt',$$

$$(4.3) \quad \Phi_2(u, v) = \psi_1(t)U_a(t)v_0 - i\psi_T(t) \int_0^t U_a(t-t') \left\{ \alpha v(t') - \frac{a}{2} (u^2)(t') \right\} dt'.$$

We will choose  $\mu < \mu(\kappa, s)$  where  $d = \frac{1}{2} - 2\mu(\kappa, s)$  and  $b = \frac{1}{2} + \mu(\kappa, s)$  satisfy the conditions of Propositions 1 and 2.

According to Lemma 1, with  $b' = -d$  and Propositions 1 and 2, we have

$$\begin{aligned} \|\Phi_1(u, v)\|_{X^{\kappa, \frac{1}{2}+\mu}} &\leq c_0 \|u_0\|_{H^\kappa} + c_1 T^\mu \left( \theta \|u\|_{X^{\kappa, -\frac{1}{2}+2\mu}} + \|\bar{u}v\|_{X^{\kappa, -\frac{1}{2}+2\mu}} \right) \\ &\leq c_0 \|u_0\|_{H^\kappa} + c_1 T^\mu \left( \theta \|u\|_{X^{\kappa, \frac{1}{2}+\mu}} + \|u\|_{X^{\kappa, \frac{1}{2}+\mu}} \|v\|_{X_a^{s, \frac{1}{2}+\mu}} \right) \\ &\leq c_0 \|u_0\|_{H^\kappa} + c_1 T^\mu \left( \theta M_1 + M_1 M_2 \right), \end{aligned}$$

$$\begin{aligned} \|\Phi_2(u, v)\|_{X_a^{s, \frac{1}{2}+\mu}} &\leq c_0 \|v_0\|_{H^s} + c_2 T^\mu \left( \alpha \|v\|_{X_a^{s, -\frac{1}{2}+2\mu}} + \frac{a}{2} \|u^2\|_{X^{\kappa, -\frac{1}{2}+2\mu}} \right) \\ &\leq c_0 \|v_0\|_{H^s} + c_2 T^\mu \left( \alpha \|v\|_{X_a^{s, \frac{1}{2}+\mu}} + \frac{a}{2} \|u\|_{X^{\kappa, \frac{1}{2}+\mu}}^2 \right) \\ &\leq c_0 \|v_0\|_{H^s} + c_2 T^\mu \left( \alpha M_2 + \frac{a}{2} M_1^2 \right). \end{aligned}$$

Defining  $M_1 = 2c_0 \|u_0\|_{H^\kappa}$  and  $M_2 = 2c_0 \|v_0\|_{H^s}$ . We have that

$$\|\Phi_1(u, v)\|_{X^{\kappa, \frac{1}{2}+\mu}} \leq \frac{M_1}{2} + c_1 T^\mu \left( \theta M_1 + M_1 M_2 \right)$$

and

$$\|\Phi_2(u, v)\|_{X_a^{s, \frac{1}{2}+\mu}} \leq \frac{M_2}{2} + c_2 T^\mu \left( \alpha M_2 + \frac{a}{2} M_1^2 \right).$$

Then  $(\Phi_1(u, v), \Phi_2(u, v)) \in \Sigma_\mu$  for

$$(4.4) \quad T^\mu \leq \frac{1}{2} \min \left\{ \frac{1}{c_1(\theta + M_2)}, \frac{M_2}{c_2(\alpha M_2 + \frac{a}{2} M_1^2)} \right\}.$$

Similarly, we such that

$$\|\Phi_1(u, v) - \Phi_1(\tilde{u}, \tilde{v})\|_{X^{\kappa, \frac{1}{2}+\mu}} \leq c_3(M_1, M_2) T^\mu \left( \|u - \tilde{u}\|_{X^{\kappa, \frac{1}{2}+\mu}} + \|v - \tilde{v}\|_{X_a^{s, \frac{1}{2}+\mu}} \right),$$

$$\|\Phi_2(u, v) - \Phi_2(\tilde{u}, \tilde{v})\|_{X_a^{s, \frac{1}{2}+\mu}} \leq c_4(M_1, M_2) T^\mu \left( \|u - \tilde{u}\|_{X^{\kappa, \frac{1}{2}+\mu}} + \|v - \tilde{v}\|_{X_a^{s, \frac{1}{2}+\mu}} \right).$$

That said, it follows that

$$(4.5) \quad \left\| \left( \Phi_1(u, v), \Phi_2(u, v) \right) - \left( \Phi_1(\tilde{u}, \tilde{v}), \Phi_2(\tilde{u}, \tilde{v}) \right) \right\|_{\Sigma_\mu} \leq \frac{1}{2} \|(u, v) - (\tilde{u}, \tilde{v})\|_{\Sigma_\mu}.$$

to

$$T^\mu \leq \frac{1}{4} \min \left\{ \frac{1}{c_3(M_1, M_2)}, \frac{1}{c_4(M_1, M_2)} \right\}.$$

Therefore, the map  $\Phi_1 \times \Phi_2 : \Sigma_\mu \rightarrow \Sigma_\mu$  is a contraction, and via the Fixed Point Theorem there is a unique solution to the Cauchy Problem for  $T$  satisfying (4.4) and (4.5).

□

**Remark 4.** *The case  $p = q = -1$  can be treated by using the same ideas that in the case  $p = q = 1$ , for any  $\sigma > 0$ .*

**Remark 5.** *The case  $p = -1$  and  $q = 1$  or  $p = 1$  and  $q = -1$  (for all  $\sigma > 0$ ) is the same in the case  $p = q = 1$  for  $\sigma > 2$ .*

## 5 Global well-posedness results

In this section we will study the global well-posedness for the system (5.1) below:



$$(5.1) \quad \begin{cases} i\partial_t u + p\partial_x^2 u - \theta u + \bar{u}v = 0 \\ i\sigma\partial_t v + q\partial_x^2 v - \alpha v + \frac{1}{2}u^2 = 0, & t \in [-T, T], x \in \mathbb{R}, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & (u_0, v_0) \in H^\kappa(\mathbb{R}) \times H^s(\mathbb{R}), \end{cases}$$

where  $u$  and  $v$  are complex valued functions.

One of the interests in working with system of equations in physics is to obtain pulse stability for certain types of solutions. In this case, it is essential to have Global well-posed results.

Starting from the Conservation Law

$$(5.2) \quad E(u, v)(t) = \|u\|_{L^2}^2 + 2\sigma \|v\|_{L^2}^2,$$

it is known that if  $u$  and  $v$  are solutions of this system with initial conditions  $(u_0, v_0) \in L^2 \times L^2$ , then  $\forall t \in \mathbb{R}$  such that  $E(u, v)(t) = E(u, v)(0) = \|u_0\|_{L^2}^2 + 2\sigma \|v_0\|_{L^2}^2$ .

Our main result presented here is Theorem 2.

To get the above result we will follow the ideas presented in [4], [7], [16] and [8].

We note here that we did not explore the second quantity conserved for light regularities, for example, greater than 1, i.e.,

$$(5.3) \quad H(u, v)(t) = p \|u_x\|_{L^2}^2 + q \|v_x\|_{L^2}^2 + \theta \|u\|_{L^2}^2 + \alpha \|v\|_{L^2}^2 - \operatorname{Re}\langle u^2, \bar{v} \rangle_{L^2}.$$

## 5.1 Preliminary results

This section is devoted to the proof of the global well-posedness result stated in Theorem 2 via the **I**-method.

Let  $s \leq 0$  and  $N > 1$  be fixed. Let's define the Fourier multiplier operator

$$(5.4) \quad \widehat{I_N^{-s}u}(\xi) = \widehat{Iu}(\xi) = m(\xi)\widehat{u}(\xi), \quad m(\xi) = \begin{cases} 1, & |\xi| < N, \\ N^{-s}|\xi|^s, & |\xi| \geq 2N \end{cases}$$

where  $m$  is a smooth non-negative function.

**Lemma 5.** *The operator  $I$  applies  $H^s(\mathbb{R}) \mapsto L^2$ . Moreover, the operator  $I$  commute with differential operators and  $Iu = I\bar{u}$ . That is,*

$$1. \|I(u)\|_{L^2} \leq cN^{-s} \|u\|_{H^s}$$

$$2. P(D)I(u) = I(P(D)u),$$

where  $P$  is a polynomial and  $D = \frac{d}{idx}$  is the differential operator.

*Proof.* It follows from the definition of  $I$  and properties of the Fourier Transform.  $\square$

We will need the following

**Lemma 6** (Lemma 12.1 of [6]). *Let  $\alpha_0 > 0$  and  $n \geq 1$ . Suppose  $Z, X_1, \dots, X_n$  are translation-invariant Banach spaces and  $T$  is a translation invariant  $n$ -linear operator such that*

$$\|I_1^\alpha T(u_1, \dots, u_n)\|_Z \leq c \prod_{j=1}^n \|I_1^\alpha u_j\|_{X_j},$$

for all  $u_1, \dots, u_n, 0 \leq \alpha \leq \alpha_0$ . Then,

$$\|I_N^\alpha T(u_1, \dots, u_n)\|_Z \leq c \prod_{j=1}^n \|I_N^\alpha u_j\|_{X_j},$$

for all  $u_1, \dots, u_n, 0 \leq \alpha \leq \alpha_0$  and  $N \geq 1$ . Here, the implied constant is independent of  $N$ .

Another essential result is

**Lemma 7** (Lemma 5.1 of [8]). *We have*

$$\left\| \left( D_x^{1/2} f \right) \cdot g \right\|_{L_{x,t}^2} \leq c \|f\|_{X^{0,1/2}} \|g\|_{X^{0,1/2}},$$

if  $|\xi_2| \ll |\xi_1|$  for any  $|\xi_1| \in \text{supp}(\widehat{f})$  and  $|\xi_2| \in \text{supp}(\widehat{g})$ . Moreover, this estimate is true if  $f$  and/or  $g$  is replaced by its complex conjugate in the left-hand side of the inequality.

**Remark 6.** The lemma above is valid replacing  $X^{0,1/2}$  by  $X_a^{0,1/2}$ .

## 5.2 Local well-posedness revisited

In the continuation, we take  $N \gg 1$  one large integer and we denote by  $I$  the operator  $I := I_N^{-s}$  for a given  $s \in \mathbb{R}$ .

We have that the system (5.1) applied to the operator  $I$  is given by

$$(5.5) \quad \begin{cases} i\partial_t Iu + p\partial_x^2 Iu - \theta Iu + I(\overline{u}v) & = 0 \\ i\sigma\partial_t Iv + q\partial_x^2 Iv - \alpha Iv + \frac{1}{2}I(u^2) & = 0 \end{cases}.$$

Let's cite here a lemma that will be used to demonstrate the local well-posedness theorem, and then reworking the bilinear estimates.

**Lemma 8.** *Given  $-1/2 < b' \leq b < 1/2$ ,  $s \in \mathbb{R}$ ,  $a \geq 0$  and  $0 < T < 1$  such that*

$$(5.6) \quad \|\psi_T(t)u\|_{X_a^{s,b'}} \leq cT^{b-b'} \|u\|_{X_a^{s,b}}.$$

*Proof.* The proof follows from article [10]. □

**Lemma 9.** Consider  $1/4 < d$ . Given  $b_1, b_2 \in \mathbb{R}$  such that  $(b_1, b_2) = \left(0, \frac{1}{2} +\right)$  or  $(b_1, b_2) = \left(\frac{1}{2} +, 0\right)$ . Then

$$(5.7) \quad \|\bar{u} \cdot v\|_{X^{0,-d}} \leq c \|u\|_{X^{0,b_1}} \cdot \|v\|_{X_a^{0,b_2}}.$$

*Proof.* Without loss of generality, let us prove only the case  $b_2 = 0$  and  $b_1 = \frac{1}{2} +$ . Following the ideas from Proposition 1, we such that

$$\|\bar{u} \cdot v\|_{X^{0,-d}} \leq \|u\|_{X^{0,b_1}} \|v\|_{X^{0,b_1}} \left\| \frac{1}{\langle \tau_2 + a\xi_2^2 \rangle^{2b_2}} \int_{\mathbb{R}^2} \frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{2b_1} \langle \tau + \xi^2 \rangle^{2d}} d\xi d\tau \right\|_{L_{\xi_2, \tau_2}^\infty}.$$

There remains the bounded of the last factor on the right side of the inequality above. Using the Lemma 2.7 and the Lemma 2.8, we have that

$$\int_{\mathbb{R}^2} \frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{2b_1} \langle \tau + \xi^2 \rangle^{2d}} d\xi d\tau \leq \int_{\mathbb{R}^2} \frac{1}{\langle \tau_2 + 2\xi^2 + \xi_2^2 - 2\xi\xi_2 \rangle^{2d}} d\xi d\tau \leq c.$$

□

In an analogous way, we prove the lemma below.

**Lemma 10.** Consider  $1/4 < d$ . Given  $b_1, b_2 \in \mathbb{R}$  such that  $(b_1, b_2) = \left(0, \frac{1}{2} +\right)$  or  $(b_1, b_2) = \left(\frac{1}{2} +, 0\right)$ . Then

$$(5.8) \quad \|uw\|_{X_a^{0,-d}} \leq c \|u\|_{X^{0,b_1}} \cdot \|w\|_{X^{0,b_2}}.$$

**Remark 7.** The above results are independent of the value of  $a > 0$ .

Now let's revisit the fixed-point theorem to find the best exponent for  $\delta$ .

**Proposition 7.** For all  $(u_0, v_0) \in H^s \times H^s$  and  $s \geq -\frac{1}{4}$  and  $0 < a < \frac{1}{2}$  or  $s \geq -\frac{1}{2}$  and  $a > \frac{1}{2}$  the system (5.5) has a unique local-in-time solution  $(u(t), v(t))$  defined on the time interval  $[0, \delta]$  for some  $\delta \leq 1$  satisfying

$$(5.9) \quad \delta \sim \left( \|Iu_0\|_{L_x^2} + \|Iv_0\|_{L_x^2} \right)^{-\frac{4}{3}+}.$$

Furthermore,  $\|Iu_0\|_{X^{0,1/2+}} + \|Iv_0\|_{X_a^{0,1/2+}} \leq c(\|Iu_0\|_{L^2} + \|Iv_0\|_{L^2})$ .

*Proof.* Using the Lemmas 5-10 the proof follows in a similar way the Proposition 5.5 of [8]. □

### 5.3 Almost conservation of the modified energy

Let us consider the energy  $E$  associated with the system(5.5)

$$(5.10) \quad E(Iu, Iv) = \|Iu\|_{L^2}^2 + 2\sigma \|Iv\|_{L^2}^2.$$

**Theorem 3.** *The functional energy (5.10) has derived with respect to the time given by:*

$$\frac{d}{dt} E(Iu, Iv) = 2\text{Im} \left\{ \int (I(\overline{u}v) - I\overline{u}Iv) I\overline{u} dx \right\} + 2\text{Im} \left\{ \int (I(u^2) - (Iu)^2) I\overline{u} dx \right\}.$$

*Proof.* Also using the following fact  $\int \overline{f} \cdot \partial_x^2 f = \int |\partial_x f|^2$ , we get:

$$\begin{aligned} \frac{d}{dt} E(Iu, Iv) &= \int \partial_t Iu \cdot I\overline{u} + \int Iu \cdot \partial_t I\overline{u} + 2\sigma \int \partial_t Iv \cdot I\overline{v} + 2\sigma \int Iv \cdot \partial_t I\overline{v} \\ &= -2\text{Im} \left\{ \int (I(\overline{u}v) - I\overline{u}Iv) \cdot I\overline{u} \right\} + 2\text{Im} \left\{ \int (I(\overline{u}^2) - (I\overline{u})^2) \cdot Iv \right\}. \end{aligned}$$

□

We know that the time of existence  $\delta = (\|Iu\|_{L^2} + \|Iv\|_{L^2})^{-4/3}$ .

Let's now estimate the modified energy. Using the fundamental theorem of calculus we such that

$$\begin{aligned} E(Iu, Iv)(\delta) - E(Iu, Iv)(0) &= 2\text{Im} \int_0^\delta \left( \int (I(\overline{u}v) - I\overline{u}Iv) I\overline{u} dx \right) dt \\ &= 2\text{Im} \int_0^\delta \left\langle (I(\overline{u}v) - I\overline{u}Iv)^\wedge; \widehat{Iu} \right\rangle_{L^2} dt \\ &\quad + 2\text{Im} \int_0^\delta \left\langle (I(u^2) - (Iu)^2)^\wedge; \widehat{Iv} \right\rangle_{L^2} dt. \end{aligned}$$

Observe that

$$\begin{aligned} (I(\overline{u}v) - I\overline{u}Iv)^\wedge &= m(\xi) \widehat{\overline{u} \cdot v} - \widehat{I\overline{u}} * \widehat{Iv} \\ &= \int \widehat{I\overline{u}}(\xi_1) \widehat{Iv}(\xi_2) \left( \frac{m(\xi) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} \right) d\xi_1. \end{aligned}$$

We such that

$$\begin{aligned} (I(u^2) - (Iu)^2)^\wedge &= m(\xi) \widehat{u^2} - \widehat{Iu} * \widehat{Iu} \\ &= \int \widehat{Iu}(\xi_1) \widehat{Iu}(\xi_2) \left( \frac{m(\xi) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} \right) d\xi_1. \end{aligned}$$

Therefore,

$$\int_0^\delta \left\langle (I(\overline{u}v) - I\overline{u}Iv)^\wedge; \widehat{Iu} \right\rangle_{L^2} dt = \int_0^\delta \int_{\mathbb{R}_\xi} \int_{\mathbb{R}_{\xi_1}} \widehat{I\overline{u}}(\xi_1) \widehat{Iv}(\xi_2) \widehat{I\overline{u}}(\xi) M(\xi, \xi_1) d\xi_1 d\xi dt,$$

analogously we have that

$$\int_0^\delta \left\langle (I(u^2) - (Iu)^2)^\wedge; \widehat{Iv} \right\rangle_{L^2} dt = \int_0^\delta \int_{\mathbb{R}_\xi} \int_{\mathbb{R}_{\xi_1}} \widehat{Iu}(\xi_1) \widehat{Iu}(\xi_2) \widehat{I\overline{v}}(\xi) M(\xi, \xi_1) d\xi_1 d\xi dt,$$

$$\text{where } M(\xi, \xi_1) = \left( \frac{m(\xi) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)} \right).$$

We note that fixed  $N > 1$ ,  $|\xi_1| \sim N_1$  and  $|\xi_2| \sim N_2$  such that:

- (i) If  $2|\xi_1| \leq |\xi_2|$  and  $2|\xi_1| \leq N$  then  $|M(\xi, \xi_1)| \lesssim \frac{N_1}{N_2}$ ;
- (ii) If  $2|\xi_2| \leq |\xi_1|$  and  $2|\xi_2| \leq N$  then  $|M(\xi, \xi_1)| \lesssim \frac{N_2}{N_1}$ ;
- (iii) If  $2|\xi_1| \leq |\xi_2|$  and  $|\xi_1| \geq 2N$  then  $|M(\xi, \xi_1)| \lesssim \frac{N_1}{N}$ ;
- (iv) If  $2|\xi_2| \leq |\xi_1|$  and  $|\xi_2| \geq 2N$  then  $|M(\xi, \xi_1)| \lesssim \frac{N_2}{N}$  and
- (v) If  $|\xi_1| \sim |\xi_2| \gtrsim N$  then  $|M(\xi, \xi_1)| \lesssim \left( \frac{N_1}{N} \right)^2$ .

By the symmetry of the variables it is sufficient to verify only the statements (i), (iii) and (v).

We will use the fact that  $m'(\xi) = -N|\xi|^{-2}$ .

In the first case, as  $|\xi_1| \ll N$  such that  $m(\xi_1) = 1$ , hence,

$$|M(\xi, \xi_1)| = \left| \frac{m(\xi_1 + \xi_2) - m(\xi_2)}{m(\xi_2)} \right| \sim \left| \frac{m'(\xi_2)|\xi_1|}{m(\xi_2)} \right| \lesssim \frac{N_1}{N_2}.$$

Still to verify the item (iii) we need to observe before that  $\frac{1}{2}|\xi_2| \leq |\xi_1 + \xi_2| \leq 2|\xi_2|$  and thereby,

$$\begin{aligned} \frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_2)} &= \frac{N|\xi_1 + \xi_2|^{-1} - N|\xi_2|^{-1}N|\xi_1|^{-1}}{N|\xi_2|^{-1}} \\ &= \frac{|\xi_2|}{|\xi_1 + \xi_2|} - \frac{N}{|\xi_1|} \\ &\leq 2 - \frac{N}{|\xi_1|} \sim 1. \end{aligned}$$

Soon, (iii) follows easily from observation that  $M(\xi, \xi_1) \sim \frac{1}{m(\xi_1)} = \frac{N_1}{N}$ .

The last case, (v), it follows from fact that

$$\begin{aligned} m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2) &= N|\xi_1 + \xi_2|^{-1} - N^2|\xi_1|^{-1}|\xi_2|^{-1} \\ &\sim N \left( \frac{1}{2|\xi_1|} - \frac{N}{|\xi_1|^2} \right) \\ &= \frac{N}{2|\xi_1|} \frac{|\xi_1| - 2N}{|\xi_1|} \sim 1. \end{aligned}$$

Therefore,  $M(\xi, \xi_1) \sim \frac{1}{m(\xi_1)m(\xi_2)} \sim \left( \frac{N_1}{N} \right)^2$ .

Considering

$$(5.11) \quad L_1 = 2\text{Im} \int_0^\delta \int_{\mathbb{R}_\xi} \int_{\mathbb{R}_{\xi_1}} \widehat{I\bar{u}}(\xi_1) \widehat{Iv}(\xi_2) \widehat{I\bar{u}}(\xi) M(\xi, \xi_1) d\xi_1 d\xi dt$$

and

$$(5.12) \quad L_2 = 2\text{Im} \int_0^\delta \int_{\mathbb{R}_\xi} \int_{\mathbb{R}_{\xi_1}} \widehat{Iu}(\xi_1) \widehat{Iu}(\xi_2) \widehat{I\bar{v}}(\xi) M(\xi, \xi_1) d\xi_1 d\xi dt,$$

we get,

$$|E(Iu, Iv)(\delta) - E(Iu, Iv)(0)| = |L_1 + L_2|.$$

**Proposition 8.** *For  $\sigma > 2$  and  $s \geq -1/2$  we have*

$$(5.13) \quad |E(Iu, Iv)(\delta) - E(Iu, Iv)(0)| \leq N^{-\frac{1}{2}} \delta^{\frac{1}{2}} \|I(u)\|_{X^{0, \frac{1}{2}+}}^2 \|I(v)\|_{X^{0, \frac{1}{2}+}}.$$

*Proof.* It is enough to estimate  $L_1$  and  $L_2$ . We still note that  $L_1$  and  $L_2$  are equivalent. In this case, let's restrict ourselves to estimating  $L_1$ . Let's use the notation  $|\xi| = |\xi_1 + \xi_2| \sim N_3$

For  $2|\xi_1| \leq |\xi_2|$  and  $2|\xi_1| \leq N$  such that  $|M(\xi, \xi_1)| \lesssim \frac{N_1}{N_2}$ . Then, from Lemmas 7 and 8, we see that

$$\begin{aligned} |L_1| &\leq \left( \frac{N_1}{N_2} \right)^{1/2} \left\| D_x^{1/2} \widehat{I\bar{u}}(\xi_1) \cdot \widehat{Iv}(\xi_2) \right\|_{L^2} \left\| \widehat{I\bar{u}} \right\|_{L^2} \\ &\leq \left( \frac{N_1}{N_2} \right)^{1/2} N_3^{-1/2} \left\| \widehat{Iu} \right\|_{X^{0, 1/2+}} \left\| \widehat{Iv} \right\|_{X^{0, 1/2+}} \delta^{1/2} \left\| \widehat{Iu} \right\|_{X^{0, 1/2+}} \\ &\leq N^{-1/2} \delta^{1/2} \|I(u)\|_{X^{0, \frac{1}{2}+}}^2 \|I(v)\|_{X^{0, \frac{1}{2}+}}. \end{aligned}$$

The case (ii), this is,  $2|\xi_2| \leq |\xi_1|$  and  $2|\xi_2| \leq N$  follows by the symmetry of the variables.

In the prove of cases (iii) and (iv) when  $s = -1/2$ , such that  $|M(\xi, \xi_1)| \lesssim \left(\frac{N_1}{N}\right)^{1/2}$

$$\begin{aligned}
|L_1| &\leq \left(\frac{N_1}{N}\right)^{1/2} \left\| D_x^{1/2} \widehat{Iu}(\xi_1) \cdot \widehat{Iv}(\xi_2) \right\|_{L^2} \left\| \widehat{Iu} \right\|_{L^2} \\
&\leq \left(\frac{N_1}{N}\right)^{1/2} N_3^{-1/2} \left\| \widehat{Iu} \right\|_{X^{0,1/2+}} \left\| \widehat{Iv} \right\|_{X^{0,1/2+}} \delta^{1/2} \left\| \widehat{Iu} \right\|_{X^{0,1/2+}} \\
&\leq N^{-1/2} \delta^{1/2} \|I(u)\|_{X^{0, \frac{1}{2}+}}^2 \|I(v)\|_{X^{0, \frac{1}{2}+}}.
\end{aligned}$$

For the last case, we have  $|M(\xi, \xi_1)| \lesssim \frac{N_1}{N}$  when  $|\xi_1| \sim |\xi_2| \gtrsim N$  thereby,  $|\xi_1| \leq 2|\xi|$  and thereby

$$\begin{aligned}
|L_1| &\leq \frac{N_1}{N} \left\| D_x^{1/2} \widehat{Iu}(\xi_1) \cdot \widehat{Iv}(\xi_2) \right\|_{L^2} \left\| \widehat{Iu} \right\|_{L^2} \\
&\leq \frac{N_1}{N} N_1^{-1/2} \left\| \widehat{Iu} \right\|_{X^{0,1/2+}} \left\| \widehat{Iv} \right\|_{X^{0,1/2+}} \delta^{1/2} \left\| \widehat{Iu} \right\|_{X^{0,1/2+}} \\
&\leq N^{-1} \delta^{1/2} \|I(u)\|_{X^{0, \frac{1}{2}+}}^2 \|I(v)\|_{X^{0, \frac{1}{2}+}}.
\end{aligned}$$

Finishing the proof.  $\square$

Following the same arguments presented above, we prove the following

**Proposition 9.** *For  $0 < \sigma < 2$  and  $s \geq -1/4$  have*

$$(5.14) \quad |E(Iu, Iv)(\delta) - E(Iu, Iv)(0)| \leq N^{-\frac{1}{4}} \delta^{\frac{1}{2}} \|I(u)\|_{X^{0, \frac{1}{2}+}}^2 \|I(v)\|_{X^{0, \frac{1}{2}+}}.$$

*Proof.* Analogous to above.  $\square$

## 5.4 Global existence

In this subsection we will demonstrate Theorem 2.

*Proof.* Given the initial conditions of the Cauchy Problem (5.1)  $(u_0, v_0) \in H^s \times H^s$  such that

$$\|I(u_0)\|_{L^2} \leq cN^{-s} \|\cdot\|_{H^s} \quad \text{and} \quad \|I(v_0)\|_{L^2} \leq cN^{-s} \|v_0\|_{H^s}.$$

Applying the local well-posedness result of the Proposition 7, we see that There is a unique solution in the time interval  $[0, \delta]$ , where  $\delta \sim N^{-4s/3}$  and such that

$$\|I(u)\|_{X^{0, \frac{1}{2}+}} + \|I(v)\|_{X^{0, \frac{1}{2}+}} \leq cN^{-s}.$$

For  $\sigma > 2$  and  $s \geq -\frac{1}{2}$  and using the Proposition 8, we have

$$|E(Iu, Iv)(\delta) - E(Iu, Iv)(0)| \leq N^{-\frac{1}{2}} \delta^{\frac{1}{2}} N^{-3s}.$$

We should now prove that for every  $T > 0$  we can extend our solution to the range  $[0, T]$ , for this it is enough to apply the local well-posedness Theorem 7 until we reach this interval, that is,  $T/\delta$  times. If the modified energy does not grow more than the initial one for this number of interactions we can conclude that the result is extended up the interval  $[0, T]$ , that is, we should have

$$(5.15) \quad |E(Iu, Iv)(\delta) - E(Iu, Iv)(0)| \frac{T}{\delta} \ll E(Iu_0, Iv_0).$$

Therefore, it is sufficient that

$$(5.16) \quad N^{-\frac{1}{2}} \delta^{\frac{1}{2}} N^{-3s} \frac{T}{\delta} \ll N^{-2s} \quad \text{or} \quad N^{-\frac{1}{2}} \delta^{-\frac{1}{2}} N^{-3s} T \ll N^{-2s}.$$

Hence we conclude that  $-\frac{1}{2} - 3s + \frac{2s}{3} \leq -2s$  because  $\delta^{-1/2} \sim N^{2s/3}$ . It turns out that for any  $s \geq -1/2$  we can extend the solution at any time interval by taking  $1 \ll N$ .

The prove of the other case ( $0 < \sigma < 2$ ) follows similarly. □

The Theorem showed in this section tells us that the solution of the Cauchy Problem extends globally, in time, in the sense that it connects the points  $(0, 0)$  and  $(-1/2, -1/2)$  when  $\sigma > 2$  and the points  $(0, 0)$  and  $(-1/4, -1/4)$  in the case  $0 < \sigma < 2$ .

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